

Solutions to the Unsolved Problems in Gelfand's *Algebra*

Adrian S. Durham

2007

Here are my solutions to as well as brief explanations of some of the problems in *Algebra* by I. M. Gelfand and A. Shen. Some problems worth noting are 166 to prove that a polynomial of degree n is determined by $n + 1$ of its values, problems 266 & 227 to prove that the square roots of two and three are irrational, problem 300 to prove the essence of the fact that the Harmonic Series diverges, problem 323 to prove the arithmetic mean is greater than the geometric mean, and problem 342 to prove that the arithmetic mean is greater than the geometric mean is greater than the harmonic mean.

Copyright:

I am putting this document in the public domain.

Acknowledgments:

A number of people have emailed suggests and corrections, many of whom use anonymous email addresses, so I will not name them here. But, I hope they will accept my collective thanks for their contribution.

Problem 3:

The boy just has to copy the digits of the three digit number down twice in sequence to get the resulting product: 715,715. The first one in 1,001 moves the digits of the three digit number three to the left in the resulting product so that they do not have to be added to the digits created by the second one in the 1,001.

Problem 4:

Again, you can just copy fifty-seven down in sequence several times to get the answer: 5,757,575,757. The ones in 101,010,101 are placed so that they would put the digits of any two digit number into spots in the resulting product where they don't have to be added to any other digits, so you can just start copying them down in sequence to get the product. But, even if you didn't do that in this case, you should quickly see that it works out that way as you carry out the addition. Here is the stack created when multiplying the numbers:

$$\begin{array}{r}
 1\ 01\ 01\ 01\ 01 \\
 \times\ 57 \\
 \hline
 57 \\
 57\ 00 \\
 57\ 00\ 00 \\
 57\ 00\ 00\ 00 \\
 57\ 00\ 00\ 00\ 00 \\
 \hline
 57\ 57\ 57\ 57\ 57
 \end{array}$$

Problem 5:

Similar to the previous problems, the placement of ones and zeros makes the addition and multiplication easier than it might otherwise be, and more importantly, there is a pattern to it. I think, in this case, you are probably just supposed to carry out the multiplication as opposed to being able to spot a trick. Here is the stack:

$$\begin{array}{r}
 10\ 20\ 30\ 40\ 50 \\
 \times\ 1\ 00\ 01 \\
 \hline
 10\ 20\ 30\ 40\ 50 \\
 10\ 20\ 30\ 40\ 50\ 00\ 00 \\
 \hline
 10\ 20\ 40\ 60\ 80\ 40\ 50
 \end{array}$$

Problem 6:

While this problem has no zeros in it to make the addition easy, its regularity does, and, in any case, there is a pattern that emerges. Again, I think the student is maybe less expected to "see the trick" rather than to just do the problem and see the pattern that emerges after the fact. The important thing is to see the pattern and hopefully understand why it happens.

$$\begin{array}{r}
 1\ 1111 \\
 \times\ 1111 \\
 \hline
 1\ 1111 \\
 11\ 1110 \\
 111\ 1100 \\
 1111\ 1000 \\
 \hline
 1234\ 4321
 \end{array}$$

As one keeps getting shifted out of the final sum in each column of the multiplication. Thus, only in the middle does each one of the 11,111 contribute to the digit in the final product while fewer and fewer ones from the 11,111 can contribute to the digits that are on either end of the final product.

Problem 8:

Here again, we are solving a little math riddle that isn't just a random addition problem but one whose answer follows some pattern that might help us experience some of the structure of arithmetic. The divisor goes into chunks of the dividend with no remainder if you actually perform the division using the long division algorithm. But, one might readily enough observe, after doing the preceding problems, that you get the dividend by multiplying the divisor by some number with ones and zeros so as to shift the digits from the divisor in a manner that they do not have to actually be added to one another. You will need to shift the three digit divisor over three places each time, and since it is repeated in the dividend three times, you will need to do this shift twice (or three times if you count not shifting at all as the first shifting of the digits). So, with this in mind, let's do the analysis in previous problems backwards. First, let's recall the vertical stack at the end of the problems of previous multiplications we have done and how it must look:

$$\begin{array}{r}
 \quad \quad ? \ ? \ ? \\
 \times \ 123 \\
 \hline
 \quad \quad 123 \\
 \quad 123 \ 000 \\
 123 \ 000 \ 000 \\
 \hline
 123 \ 123 \ 123
 \end{array}$$

What multiplication problem gives us this stack? Our answer is $??? = 1,001,001$. To find it directly we can use the division algorithm as follows:

$$\begin{array}{r}
 \quad \quad 1 \ 001 \ 001 \\
 123 \overline{) 123 \ 123 \ 123} \\
 \underline{123} \\
 \quad 0 \ 123 \\
 \quad \underline{123} \\
 \quad \quad 0 \ 123 \\
 \quad \quad \underline{123} \\
 \quad \quad \quad 0
 \end{array}$$

Problem 9:

Following problem 8, the divisor goes into chunks of the dividend with no remainder if you actually perform the division using the long division algorithm. But, again following the pattern of the preceding multiplication problems (and problem 8), you get the dividend by multiplying the divisor by some number with ones and zeros so as to shift the digits from the divisor in a manner that they do not have to actually be added to one another. You will need to shift the now seven digit divisor over (this time) *seven* each time. So, we will generally have a number consisting of ones separated by six zeros that we multiply by 1,111,111 to get the one hundred ones of the dividend.

However, in this case there will be a remainder which means that the quotient won't give us all one hundred ones of the dividend. If we imagine actually carrying out the long division, we might see that the quotient will multiply by the divisor to give you all the leading ones in batches of seven. (The quotient will be something like "100000010000001" The first one in the quotient gets you the first seven ones in the dividend when you multiply the quotient by 1,111,111. The next one in the quotient gets you the next seven ones of the dividend. And, so on.) So, there will be no remainder as long as seven evenly divides the number of ones in the dividend. And, the remainder will be the result of seven not evenly going into the number of ones in the dividend.

In particular, for a dividend consisting of one hundred ones, we get a quotient with fourteen ones in it each separated by six zeros and with a trailing two zeros (because $100 = 7 \times 14 + 2$). When we multiply the quotient with the divisor, we will get ninety-eight ones followed by two zeros which means that the remainder must be eleven to give us the one hundred ones of the divisor. To help us further visualize the arithmetic taking place, here is the stack for the multiplication of the quotient I describe above and the divisor (with some ellipses for space):

$$\begin{array}{r}
 1\ 0000001\ 0000001\ \dots\ 0000001\ 0000001\ 00 \\
 \times\ 1111111 \\
 \hline
 111111\ 00 \\
 1111111\ 0000000\ 00 \\
 \dots \\
 1111111\ \dots\ 0000000\ 0000000\ 00 \\
 1111111\ 0000000\ \dots\ 0000000\ 0000000\ 00 \\
 \hline
 1111111\ 0000000\ 0000000\ \dots\ 0000000\ 0000000\ 00 \\
 1111111\ 1111111\ 1111111\ \dots\ 1111111\ 1111111\ 00
 \end{array}$$

Problem 10:

This problem is actually partially solved in the question of the next problem. Gelfand does some of the long division to show you that $1,000,000 \div 7 = 142,857r1$ (with a remainder of one). As we'll see in problem 11, because the remainder is one and the dividend is just a one followed by zeros in both cases, the quotient will continue to repeat. In fact, every time we end up having to divide ten by seven in the division algorithm is precisely when our quotient will start repeating again. Since we require six zeros to get back to a situation of dividing ten by seven, we will repeat the quotient three times because six goes into twenty three times with two left over. And, furthermore, in examining Gelfand's long division exhibit, we can see that the last two digits of the quotient will be one and four with a remainder of two. So, our quotient is: 14,285,714,285,714r2. Or, in other words, $7 \times 14,285,714,285,714 + 2$ equals a one with twenty zeros behind it.

Problem 11:

The pattern will repeat because the periodicity starts and is caused by exactly the same thing every time. In the case of dividing by seven, the pattern starts to repeat itself precisely when you end up dividing seven into ten again. When this outcome occurs, you are essentially right back to dividing seven into a one with a bunch of zeros following it. The case of dividing the hundred ones by 1,111,111 is

even simpler – everything else is repeating, so how could the quotient do anything but repeat? The next digit in the quotient has to be zero until we have accumulated enough (seven) ones, in which case, it will always be exactly one and then return to zero on the digit after that.

One interesting difference, though, between the two problems is what happens after the decimal. In both cases, the divisor does not evenly divide the dividend. In the case of dividing by seven, the pattern will continue to repeat beyond the decimal since nothing changes at the decimal in that particular problem. On the other hand, in the case of the one hundred ones, the ones stop at the decimal and only zeros continue after that. At that point, we will start to see a nontrivial set of digits popping up in the quotient rather than the repeating pattern we had been seeing.

Problem 12:

As discussed in problems 10 and 11, we will get a repeating pattern when dividing seven into these numbers. (In fact, it can be proven in general that, at most, we get an infinitely repeating pattern in the decimal expansion of any fraction or so called "rational number" if not a terminating or finite decimal expansion.) In these particular cases, the repetition will begin to occur whenever a remainder equal to the leading digit of the dividend occurs during the division algorithm.

$$\begin{array}{r}
 285714\mathbf{28} \dots \\
 7 \overline{) 2000000\mathbf{00} \dots} \\
 \underline{14} \\
 60 \\
 \underline{56} \\
 40 \\
 \underline{35} \\
 50 \\
 \underline{49} \\
 10 \\
 \underline{7} \\
 30 \\
 \underline{28} \\
 \mathbf{20} \\
 \mathbf{14} \\
 \mathbf{60} \\
 \mathbf{56} \\
 \mathbf{4} \dots
 \end{array}$$

Observe that the moment we get a remainder of two, the whole thing starts all over again in the above division. So, it goes for six zeros and then starts all over again on the seventh zero and then repeats itself for another six zeros and so on. Thus, if we have twenty zeros, we will get three repetitions of the pattern with the first two elements in the pattern left over at the end: 28,571,428r5.

If we carefully examine Gelfand's exhibit for problem 11 we can deduce that a three with a bunch of zeros divided by seven will result in a quotient with a repeating string of the same digits as the preceding two only starting at the four: 428,571. And, if the dividend starts with four, the quotient will be of the same form only starting with the five: 571,428.... Because one, two, three and four all appear as remainders somewhere in the division algorithm of the original problem, dividing seven into them will result in the same set of repeated digits only starting at the digit that corresponds to the particular remainder we use as the leading digit of our dividend. In fact, if you carefully examine Gelfand's exhibit for problem 11 you might notice that every number from one to six appear as remainders somewhere. Why might that be? At any rate, our question for this problem is answered.

Problem 13:

My wife chugged right past this problem never noticing the astonishing result! Stare at this list until the plot unravels for you:

$$\begin{array}{rcl} 1 \times 142,857 & = & 142,857 \\ 2 \times 142,857 & = & 285,714 \\ 3 \times 142,857 & = & 428,571 \\ 4 \times 142,857 & = & 571,428 \\ 5 \times 142,857 & = & 714,285 \\ 6 \times 142,857 & = & 857,142 \end{array}$$

Perhaps you noticed that each product consists of the very same digits, only in different permutations. Indeed, not just any permutation, but specifically a rotation through the digits in order starting from somewhere in the middle is what we get. This number, 142,857, is called a cyclic number. One way to see that this cyclical property must occur with multiplication is to observe that it occurs through division in the particular dividends we were looking at in the preceding problem. Those dividends are just integral products of the original dividend we looked at (a one with a bunch of zeros behind it). So, for instance, we know that a two with a bunch of zeros behind it must just be a cycle through the sequence 142,857 starting somewhere in the middle. Then, the product $2 \times 142,857$ must be a cycle through that sequence of digits since $2 \times 142,857 = 2 \times \left(\frac{1,000,000}{7} - 1 \right) = \frac{1,000,000}{7} - 2$ is a cycle of the original sequence. (Note that based on the previous problems, seven does not divide two million evenly, and in fact has a remainder of two, hence the "-2" in the last expression.) At any rate, Gelfand is probably just hoping that you notice that 142,857 is a cyclic number in this problem.

Problem 14:

These numbers aren't that easy to find by just casually doing some arithmetic; however, the next one appears to be seventeen:

$$\begin{array}{rcl} 100,000,000,000,000 & \div & 17 = 5,882,352,941,176,470 \\ 200,000,000,000,000 & \div & 17 = 1,176,470,588,235,294 \\ 300,000,000,000,000 & \div & 17 = 1,764,705,882,352,941 \end{array}$$

...

In general, the divisor will have to be prime, but just being prime is not sufficient. Without going into any further detail on this intriguing idea, Math World (mathworld.wolfram.com) has a great discussion of it and related topics.

Problem 15:

We can quickly spot what this sequence is if we translate the binary to decimal...

Binary	Decimal
0	0
1	1
10	2
11	3
100	4
101	5
110	6
111	7
1000	8
1001	9
1010	10
1011	11
1100	12
1101	13
1110	14
1111	15
10000	16
10001	17
10010	18
10011	19
10100	20
10101	21
10110	22

Problem 17:

From problem 15...

Decimal	Binary
14	1110
16	10000

Problem 18:

Well, there are at least two ways to approach this problem. One is to count up to forty-five in binary which works similarly to counting in decimal just without as many digits. You start incrementing the right most position until you run out of digits in which case the next position is incremented and the right most starts over. If you run out of digits in the next position, then increment the position to the left of that and start the next position over and so on as necessary.

Another way to approach this problem is essentially by using powers of two starting with the largest one less than forty-five. So, the "tens place" in binary is actually the twos place. The "hundreds place" is the fours place. The "thousands place" is the eights place. (We just keep doubling each time, here, so your child doesn't have to understand exponentiation in its full generality to "get it".) In the fifth spot is the sixteens place and the sixth spot – what would normally be the hundred thousands place – is the thirty-seconds. The next spot is the sixty-fours which is bigger than forty-five, so I will start my binary number with a one in the thirty-seconds spot as it must be less than 1,000,000. That is, my binary number is something like 1?????. Whatever digits I fill in these question marks with must represent a number equal to $45 - 32 = 13$ because by putting the one in the thirty-seconds spot, I am saying that I have one thirty-second (just like a one in the one hundred thousands spot means one one hundred thousand in decimal).

So, now I find the largest power of two less than thirteen which is two, four, eight, sixteen – eight. I must have a one in the thirty-seconds spot and then my next one is in the eight's spot with zeros filling everything in between. Thus, my binary number now looks like 101???. This binary number gets me one thirty-second and one eight which is a total of forty, leaving me five more to get. The biggest power of two less than five is two, four, eight – four. So, I put a one in the fours place and zeros everywhere else to get 1011??. Finally, one thirty-second, one eight and one four all total forty-four, leaving only one more to get me to forty-five which can be accomplished by making everything else zero except for the ones spot which will be one. And so, forty-five in decimal is written in binary as 101101.

Problem 19:

Binary works just like decimal only instead of using powers of ten we just double each time to get to the next spot in the binary expansion. So, the first digit is the ones spot and if there is a one there, we count a one otherwise we count nothing. The next spot is the twos spot – one doubled – so that if there is a one *there* we count two and nothing if it is zero. (There can be only a one or a zero.) And, we continue the process through the entire binary expansion to see that 10101101 represents $1 + 0 \times 2 + 4 + 8 + 0 \times 16 + 32 + 0 \times 64 + 128 = 45 + 128 = 173$ in decimal. (Also,

using the previous problem we can note that $10101101 = 10000000 + 101101$ to help simplify our calculations.)

Problem 20:

These are just straight-up addition problems. Remember that anything greater than a one in a given position requires you "carrying the one" to the next position.

Binary		Decimal
$1010 + 101 = 1111$	$(8+2) + (4+1) = (8+4+2+1)$	$10 + 5 = 15$
$1111 + 1 = 10000$	$(8+4+2+1) + (1) = (16)$	$15 + 1 = 16$
$1011 + 1 = 1100$	$(8+2+1) + (1) = (8+4)$	$11 + 1 = 12$
$1111 + 1111 = 11110$	$(8+4+2+1) + (8+4+2+1) = (16+8+4+2)$	$15 + 15 = 30$

Problem 21:

These are just straight-up subtraction problems. You can borrow from the column to the left just exactly the same way you would in decimal if necessary.

Binary		Decimal
$1101 - 101 = 1000$	$(8 + 4 + 1) - (4 + 1) = (8)$	$13 - 5 = 8$
$110 - 1 = 101$	$(4 + 2) - (1) = (4 + 1)$	$6 - 1 = 5$
$1000 - 1 = 111$	$(8) - (1) = (4 + 2 + 1)$	$8 - 1 = 7$

Problem 22:

Another ordinary arithmetic problem...

Binary		Decimal
1101	8+4+1	13
$\times 1010$	8+2	$\times 10$
11010		
1101000		
10000010	128+2	130

Problem 23:

And now, a division problem to complete your primer on binary arithmetic:

Binary	Decimal
$ \begin{array}{r} 101 \\ 101 \overline{) 11\ 011} \\ \underline{101} \\ 111 \\ \underline{101} \\ 10 \end{array} $	$ \begin{array}{r} 5 \\ 5 \overline{) 27} \\ \underline{25} \\ 2 \end{array} $

Problem 24:

Binary	Decimal
$ \begin{array}{r} 0.01\ 010\dots \\ 11 \overline{) 1.00\ 000\dots} \\ \underline{11} \\ 1\ 00 \\ \underline{11} \\ 10\dots \end{array} $	$ \begin{array}{r} 0.33\dots \\ 3 \overline{) 1.00\dots} \\ \underline{9} \\ 10 \\ \underline{9} \\ 1\dots \end{array} $

Problem 25:

I think by "try it" he means try mixing your coffee in the two different ways.

Problem 28:

$$\begin{aligned}
 899 + 1,343 + 101 &= \\
 (899 + 101) + 1,343 &= \\
 ((899 + 1) + 100) + 1,343 &= \\
 (900 + 100) + 1,343 &= \\
 1000 + 1,343 &= 2,343
 \end{aligned}$$

Problem 29:

$$37 \times 25 \times 4 = 37 \times (25 \times 4) = 37 \times 100 = 3,700$$

Problem 30:

$$\begin{aligned}
 125 \times 37 \times 8 &= \\
 37 \times (125 \times 8) &= \\
 37 \times [125 \times (4 \times 2)] &= \\
 37 \times [(125 \times 2) \times 4] &= \\
 37 \times (250 \times 4) &= \\
 37 \times 1,000 &= \\
 37,000 &
 \end{aligned}$$

Problem 31:

This one is a zinger! Assuming his example has all the ways to calculate $2 \times 3 \times 4 \times 5$, then you can know that $2 \times 3 \times 4 \times 5 \times 6$ will have all those ways with $(2 \times 3 \times 4 \times 5)$ plus all those ways again with $(3 \times 4 \times 5 \times 6)$. That is:

$((2 \times 3) \times 4) \times 5 \times 6$	$2 \times ((3 \times 4) \times 5) \times 6$
$((2 \times (3 \times 4)) \times 5) \times 6$	$2 \times ((3 \times (4 \times 5)) \times 6)$
$((2 \times 3) \times (4 \times 5)) \times 6$	$2 \times ((3 \times 4) \times (5 \times 6))$
$(2 \times ((3 \times 4) \times 5)) \times 6$	$2 \times (3 \times ((4 \times 5) \times 6))$
$(2 \times (3 \times (4 \times 5))) \times 6$	$2 \times (3 \times (4 \times (5 \times 6)))$

Now, what we are missing is the ways to put in the parentheses where the first or last two numbers are associated with each other. That is, where we are doing something like $(2 \times 3) \times (4 \times 5 \times 6)$ or $(2 \times 3 \times 4) \times (5 \times 6)$ and then subdividing from there. Again we can exploit some symmetry:

$(2 \times 3) \times ((4 \times 5) \times 6)$	$((2 \times 3) \times 4) \times (5 \times 6)$
$(2 \times 3) \times (4 \times (5 \times 6))$	$(2 \times (3 \times 4)) \times (5 \times 6)$

Now, let's take a second look at what we've done here. I've done all the combinations where the 2, 3, 4, and 5 are associated with each other and the 6 is all by itself. In the second table, we have all the combinations where the 2, 3, and 4 are together in one group and the 5 and 6 are in another. We have the symmetrical mirror image of that in the same table where 2 and 3 are in one group and 4, 5, and 6 are in another. Finally back in the first table we have the mirror image of the first grouping: 2 in a group by itself and the rest in their own group. Or in other words, we have listed out all the possibilities in each *partitioning* of $2 \times 3 \times 4 \times 5 \times 6$ into two partitions.

So, given a string of multiplications from 2 up to any number, partition this string into two pieces. For each way to do one partition, you must list out all the ways to do the other partition (which was easy for us in the above case because "the other partition" was always trivially done in one way). You do the smaller partitions the same way you do the whole – by partitioning them and listing out all the combinations. Because partitioning always reduces the size of the sequence, we are guaranteed to (eventually) be able to list all the possibilities. The above tables, then, contain all the possibilities for $2 \times 3 \times 4 \times 5 \times 6$ because they have all the partitionings and all the possibilities within each partitioning.

Problem 32:

This question is also a combinatorial question (like the last one). It is a lot more doable for a young child. The first two numbers are "free" in terms of requiring parentheses, but every number you add after that will cause you to put parentheses around some numbers to say just when that number gets multiplied. Thus, since there are 99 numbers here, you will need $99 - 2 = 97$ sets of parentheses (which, I guess, is $2 \times 97 = 194$ instances of a "(" or a ")") to completely specify the order of operations.

Or, another way to look at it is that it is always the same number of parentheses for any grouping. So, the number it will take is the same as if we do the first multiplication first, then multiply that product by the second, and so on, accumulating the product in order from left to right. The way we specify such an order is to put parentheses around the 2×3 and then parentheses around the $(2 \times 3) \times 4$ and then around the $((2 \times 3) \times 4) \times 5$ and so on. So, I stick a ")" in between each number starting between the 3 and the 4 and ending between the 99 and the 100. This will give me $99 - 2$ or $100 - 3$, depending on how you like to count it, right parentheses. And, the rest follows.

Problem 36:

Every term must be multiplied by every other term because you distribute one group to each term of the other group and then distribute a given term of the other group to each term of the first group. So, there will be $3 \times 5 = 15$ terms:

$$\begin{aligned}(a + b + c + d + e)(x + y + z) &= \\ a(x + y + z) + b(x + y + z) + c(x + y + z) + d(x + y + z) + e(x + y + z) &= \\ ax + ay + az + bx + by + bz + cx + cy + cz + dx + dy + dz + ex &+ ey + ez\end{aligned}$$

Problem 40:

Using the hint,

$$1 - \frac{10,001}{10,002} = \frac{10,002}{10,002} - \frac{10,001}{10,002} = \frac{10,002 - 10,001}{10,002} = \frac{1}{10,002}$$

And, by a similar argument

$$1 - \frac{100,001}{100,002} = \frac{100,002}{100,002} - \frac{100,001}{100,002} = \frac{100,002 - 100,001}{100,002} = \frac{1}{100,002}$$

Following the immediately preceding paragraph in Gelfand, it is better to get 1 bottle for 10,002 than it is to get one bottle for 100,002 (since $100,002 > 10,002$), so that $\frac{1}{10,002} > \frac{1}{100,002}$. Now, applying similar logic (it works kind of like a double negative), if the distance from a number greater than them is greater for one than the other, then the number that is furthest away must be smaller. (Yeah, go back and reread that one several times.) Basically, $\frac{1}{10,002}$ is greater than $\frac{1}{100,002}$, so that $\frac{10,002}{100,002}$ is further away from 1 than $\frac{100,001}{100,002}$. Since they are both less than one, being closer to one means being further from zero so that $\frac{100,001}{100,002}$ is further from zero than $\frac{10,001}{10,002}$, so that $\frac{100,001}{100,002} - 0 > \frac{10,001}{10,002} - 0$ or $\frac{100,001}{100,002} > \frac{10,001}{10,002}$.

Problem 41:

We might surmise from the last problem that if you add equal amounts to the numerator and denominator of a fraction less than one (and greater than zero), then you will get a larger number than the original fraction. I will just come out and assert that $\frac{12,346}{54,322} > \frac{12,345}{54,321}$. To check if it is true, subtract $\frac{12,345}{54,321}$ from both sides and calculate the difference:

$$\frac{12,346}{54,322} - \frac{12,345}{54,321} =$$

$$\frac{12,346}{54,322} \times \frac{54,321}{54,321} - \frac{12,345}{54,321} \times \frac{54,322}{54,322} =$$

$$\frac{(12,346 \times 54,321) - (12,345 \times 54,322)}{54,321 \times 54,322} =$$

$$\frac{12,346 \times 54,321 + 54,321 - 12,345 \times (54,321 + 1)}{54,321 \times 54,322} =$$

$$\frac{12,346 \times 54,321 + 54,321 - 12,345 \times 54,321 - 12,345}{54,321 \times 54,322} =$$

$$\frac{54,321 - 12,345}{54,321 \times 54,322} > 0$$

because $54,321 > 12,345$ (which, in general, is all that is required or that, in other words, the nonnegative fraction you are working with is less than one).

That might be a lot of calculations to follow, but just to recap, I took the difference of the two fractions and saw whether it was greater or less than zero. I did this by actually calculating the difference by finding a common denominator. Also, I spared myself a lot of computational headaches by using the distributive law. Once I determine if the difference is greater than or less than zero, then just by adding both sides by the second term of the difference, I establish which term is greater. So, because $\frac{12,346}{54,322} - \frac{12,345}{54,321}$ turns out to be greater than zero, by adding $\frac{12,345}{54,321}$ to both sides of the inequality, I see that $\frac{12,346}{54,322} - \frac{12,345}{54,321} > 0$ implies that $\frac{12,346}{54,322} > \frac{12,345}{54,321}$.

Problem 42:

Before we start this problem, let me first say that this whole problem is very difficult and it really sticks out like a sore thumb from the rest. It has typos and, originally, I even thought part (c) was false. And, it requires a great deal of formal manipulations to do, either way, which is a complete departure from the other problems. Don't be too worried if your child (or you, for that matter) wasn't able to get it.

For the parts I'm actually going to prove, I am going to make the simplifying assumption that we are dealing with non-negative fractions. If we allow negative fractions, we start getting these problems like just what is $\frac{a+c}{b+d}$ anyway? (You'll see that he makes a typo down in Part (b), below, by the way.) Does the negative go with the a or the b if $\frac{a}{b}$ is negative? For instance, consider $-\frac{1}{3}$ and $-\frac{1}{4}$. $4 \cdot (-1) - 3 \cdot (-1) = 3 - 4 = -1$, so they are neighbors. (And, this part will work out, in general, I think.) And, if I'm not being difficult, then when I do part b to it, I'll get $-\frac{2}{7}$ which is exactly what I should get. But, if I am being recalcitrant, I could say that $a = 1$, $b = -3$, $c = -1$ and $d = 4$, in which case I get $\frac{a+c}{b+d} = \frac{1+(-1)}{-2+4} = 0$ which is not even between $-\frac{1}{3}$ and $-\frac{1}{4}$! Rather than try to make sure that I formally stipulate the problem so that such outcomes cannot occur, let's just assume (without loss of generality) that we are only dealing with non-negative fractions.

- a) If $ad - bc = 1$, then $ad = bc + 1$, adding bc to both sides of the equation. Suppose that b has a factor p so that $b = p \cdot q$ for some whole number $q > 1$. Then, $ad = pqc + 1$. Suppose p

divided ad , then we could write $ad = p \cdot r$ from some whole number $r > 1$. Then, we have that $pr = pqc + 1$ or that $1 = pr - pqc = p(r - qc)$, using the distributive law. Or, dividing both sides by $r - qc$, we see that $p = \frac{1}{r - qc}$. But, p is a whole number greater than 1 and $r - qc$ is some sort of integer because r, q and c are whole numbers so that $\frac{1}{r - qc}$ cannot be a whole number. That's a contradiction, so we must reject the premise that p divides ad . Or, in particular, if a whole number greater than 1 divides b it cannot divide a , and so $\frac{a}{b}$ is in lowest terms.

The proof goes almost exactly the same way if $ad - bc = -1$. (In fact, ordinarily we would say "without loss of generality assume that $ad - bc = 1$ " and leave out the case of $ad - bc = -1$.) I am not sure how much detail we should be looking for here from a child – almost surely not as much as I have provided. The key is to see that $ad = bc + 1$ implies that there cannot be common factors between ad and bc , and, in particular, such factors cannot exist between a and b or c and d , respectively. And, the reason this implication holds is because we are adding or subtracting 1. Had we used a whole number greater than 1 (2 or more), then $\frac{a}{b}$ would *not* necessarily be reduced.

- b) First of all, I am pretty sure there is a typo. The fraction in question is not $\frac{a+b}{c+d}$ but rather $\frac{a+c}{b+d}$. For one thing, $\frac{a+b}{c+d}$ just doesn't work – try $\frac{1}{3}$ and $\frac{1}{4}$. They're neighbors $4 \cdot 1 - 3 \cdot 1 = 1$, but they aren't neighbors to $\frac{4}{5}$ ($5 \cdot 1 - 3 \cdot 4 = -7$ and $5 \cdot 1 - 4 \cdot 4 = -9$). On the other hand, they are neighbors to $\frac{2}{7}$ ($7 \cdot 1 - 3 \cdot 2 = 1$ and $7 \cdot 1 - 2 \cdot 4 = -1$). Also, in the text, he had been discussing adding numerators to numerators and denominators to denominators immediately prior to the problem. So, I think he meant to do that instead of add the numerator and the denominator of the same fraction to get the new numerator and then again from the other fraction to create the new denominator. Also, the adding of the denominators (not the numerator and denominator of one of the fractions) is where we're ultimately headed. I couldn't find it on Math World; however, neighbor fractions and combinatorial number theory is a real subject in math. (I just never studied it at all, myself.) Even correcting this typo, we encounter all sorts of problems that we need to address in the general case. So, I am really going to start using, here, the simplifying assumptions I mention above.

At any rate, let's just check to see if $\frac{a}{b}$ and $\frac{a+c}{b+d}$ are neighbor fractions. We have:

$$a(b + d) - b(a + c) = ab + ad - ba - bc = ad - bc$$

and

$$c(b + d) - d(a + c) = cb + cd - da - dc = cb - da = -(ad - bc)$$

Since $\frac{a}{b}$ and $\frac{c}{d}$ are neighbor fractions (i.e. $ad - bc$ is plus or minus one), then the preceding calculations result in plus or minus one and so $\frac{a+c}{b+d}$ is a neighbor to both $\frac{a}{b}$ and $\frac{c}{d}$. Furthermore, the numerator of one difference is just the negative of the numerator of the other difference. This fact implies that $\frac{a+c}{b+d}$ is between the two if you stare at this webpage for long enough and think about it real hard. (Normally, this is where we just boldly say "Clearly,...") Actually, if $\frac{a}{b} - \frac{a+c}{b+d}$ is positive then $\frac{c}{d} - \frac{a+c}{b+d}$ is negative, and conversely, so that: $\frac{a}{b} > \frac{a+c}{b+d}$ whenever $\frac{c}{d} < \frac{a+c}{b+d}$

and/or $\frac{a}{b} < \frac{a+c}{b+d}$ whenever $\frac{c}{d} > \frac{a+c}{b+d}$. Or, $\frac{a+c}{b+d}$ is between $\frac{a}{b}$ and $\frac{c}{d}$ (so long as they are non-negative fractions).

- c) First of all, looking back at the previous section, we know that the fraction $\frac{a+c}{b+d}$ lies in between and is a neighbor to both $\frac{a}{b}$ and $\frac{c}{d}$. Or, in particular, $\frac{a+c}{b+d} - \frac{a}{b} = \frac{1}{b(b+d)}$ and $\frac{c}{d} - \frac{a+c}{b+d} = \frac{1}{d(b+d)}$. Now what if you had a fraction $\frac{e}{f}$ so that $\frac{a}{b} < \frac{e}{f} < \frac{c}{d}$? Either $\frac{e}{f} < \frac{a+c}{b+d}$, $\frac{a+c}{b+d} < \frac{e}{f}$, or they are the same. If they are the same, then $f = b + d$ and $\frac{e}{f}$ cannot be reduced since it is a neighbor, so that f is not less than $b + d$. (In other words, what we're proving would be true in that case.) Suppose $\frac{e}{f} < \frac{a+c}{b+d}$. Then $\frac{e}{f} - \frac{a}{b} < \frac{a+c}{b+d} - \frac{a}{b}$. That is, using part (b) above, $\frac{eb-af}{fb} < \frac{1}{b(b+d)}$. Finally, multiply by $fb(b+d)$ to get that $(eb-af)(b+d) < f$. Since, $\frac{a}{b} < \frac{e}{f}$, $1 < eb-af$, and so $1 \times (b+d) < (eb-af)(b+d) < f$. It all seems so simple now... clearly....

The trick, here (for me, at any rate), was to look at the distances. The denominator required for the fraction is related to its distance from these neighbor fractions. As I recall, I was thinking in these terms originally but then must have gotten distracted somehow. Hey, if your child is thinking like this, though, I think he is born to do math. To be able to look past the formulas and see reality is a rare talent, indeed.

Problem 44:

They are exactly the same: 5% of 7,000,000,000 is

$$\begin{aligned} (5/100) \times 7,000,000,000 &= \\ (5 \times 7,000,000,000)/100 &= \\ (5 \times 7 \times 1,000,000,000)/100 &= \\ (7 \times 5 \times 1,000,000,000)/100 &= \\ (7 \times 5,000,000,000)/100 &= \\ (7/100) \times 5,000,000,000 & \end{aligned}$$

which is 7% of 5,000,000,000.

Problem 46:

- a) Well, we already know that $2^6 = 64$. We just need to multiply by 2 a few more times to get, 128, 256, 512, 1,024... $2^{10} = 1,024$.
- b) This is the decimal system! Just put the number of zeros behind a 1 that match the number displayed as the power: $10^3 = 1,000$.
- c) Again, if 10 is raised to the 7th power, then you put seven zeros behind a one: $10^7 = 10,000,000$.

Problem 47:

In decimal, each number in a sequence represents the number of some particular power of ten. So, in the number 1,234, the 2 represents 2 of 10^2 s, and you just add them up:

$$1,234 = 1 \times 10^3 + 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$$

So, to write down $10^{1,000}$ as a string of digits in decimal notation, you have none of a given power of ten except for the one of ten raised to the thousand. So, it is a 1 for that power, followed by all zeros for all the powers leading up to it, or a 1 for the $1,000^{\text{th}}$ power and zeros following the one for each of the 0^{th} through the 999^{th} power. That's one thousand zeros and one 1, or 1,001 digits.

Problem 48:

Hopefully, you learned how to do units in previous arithmetic classes (or science, even). You must do a standard conversion:

$$4 \text{ light-years} = 4 \times \left(3 \times 10^8 \frac{\text{m}}{\text{s}} \right) \times 1 \text{ year} =$$

$$4 \times 3 \times 10^8 \text{ m} \times \frac{1 \text{ year}}{1 \text{ sec}} \times \frac{365 \text{ days}}{1 \text{ year}} \times \frac{24 \text{ hours}}{1 \text{ day}} \times \frac{60 \text{ min}}{1 \text{ hour}} \times \frac{60 \text{ sec}}{1 \text{ min}}$$

Rearranging the terms, we can get the units of time to all cancel each other and end with a unitless number times some number of meters. Multiplying out the numbers gives us the distance in meters:

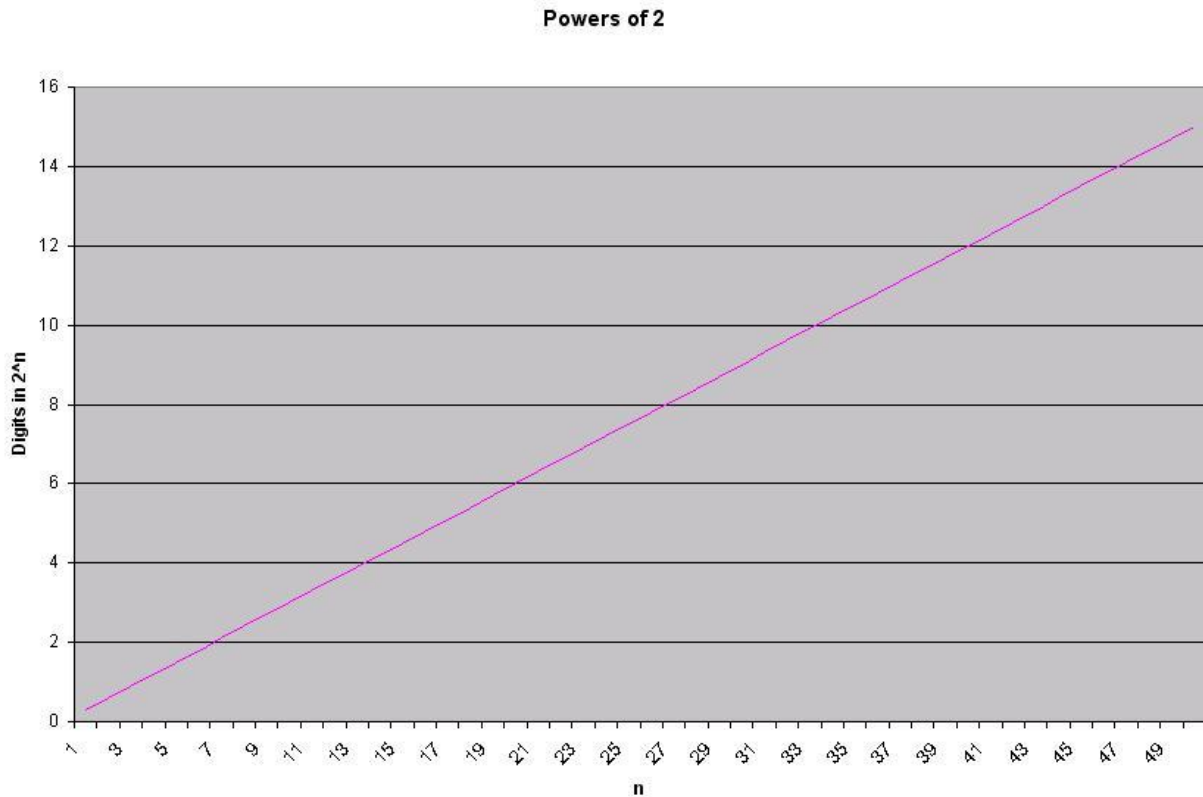
$$4 \times (3 \times 10^8 \text{ m}) \times \frac{1 \times 365 \times 24 \times 60 \times 60}{1 \times 1 \times 1 \times 1 \times 1} =$$

$$378,432,000 \times 10^8 \text{ m} = 3.78432 \times 10^{16} \text{ m}.$$

Problem 49:

- a) As we saw earlier $2^{10} = 1,024$ which is the product of 2 with itself ten times. We can multiply that product by itself to get two multiplied by itself 20 times. And so, $2^{10} = 1,024 \times 1,024 = 1,048,576$ (if we just do it long hand). So, you will need 7 digits.
- b) We'll have to be a little more clever with 2^{100} . We have $2^{10} = 1,024 = 10^3 + 24$. If we multiply this by itself 10 times to get $2^{100} = (2^{10})^{10} = (10^3 + 24)^{10}$. The 24 will certainly add to the product, but (again, through the tried and true method of just staring at the screen really hard) we can see that the powers of 10^3 are always going to be much greater than any powers of 24 and any cross terms with lower powers of 10^3 . Or, in other words, we should get the same number of digits as are in $(10^3)^{10} = 10^{30}$ which has $1 + 30 = 31$ digits.

c)



Problem 50:

Hopefully, we already know how to do this. We write a 1 with a slash and then the decimal representation of the number raised to the positive power (negate the negative).

- a) $10^{-1} = 1/10^1 = 1/10$
- b) $10^{-2} = 1/10^2 = 1/100$
- c) $10^{-3} = 1/10^3 = 1/1,000$

Problem 51:

Yes. For $n = 0$, $a^0 = 1 = \frac{1}{1} = \frac{1}{a^0} = a^{-0}$. Let's just take-in for one moment what I just did here. I did not just assume that I knew what a^{-0} meant or even that $-0 = 0$ or anything like that (although, the latter statement *is* true and I could have tried something like that, but it would only work with zero). I *use* the definition Gelfand just gave and as *directly* (and as soon) as possible. It is very common for someone with no experience proving things mathematically to encounter an assertion and have no idea what to do to prove it or even why it needs to be proven and/or whether its proof is trivial or not. **When you draw a blank** like that, looking at some statement, **return to the definitions** of the objects discussed in the statement. Be real "stupid" about it and take the definition as literally as you can.

$$\begin{aligned} a^{-n} &= a^{-(-m)} && \text{(since if } m = -n \text{ then } n = -m) \\ &= a^m \\ &= \frac{1}{1/a^m} \\ &= \frac{1}{a^{-m}} && \text{(based on Gelfand's definition of } a^{-m}) \\ &= 1/a^n \end{aligned}$$

Problem 54:

$$a^m/a^n = a^m(1/a^n) = a^m a^{-n} = a^{m+(-n)} = a^{m-n}$$

Problem 55:

- [illegible]

$$(2^{10} \times 2^{10}) \times (2^{10} \times 2^{10}) \times (2^{10} \times 2^{10}) \times (2^{10} \times 2^{10}) \times (2^{10} \times 2^{10}) \times (2^{10} \times 2^{10}) \times (2^{10} \times 2^{10}) \times 2^{10} =$$

$$2^{20} \times 2^{20} \times 2^{20} \times 2^{20} \times 2^{20} \times 2^{20} \times 2^{20} \times 2^{10} =$$

$$(2^{20} \times 2^{20}) \times (2^{20} \times 2^{20}) \times (2^{20} \times 2^{20}) \times (2^{20} \times 2^{10}) =$$

$$2^{40} \times 2^{40} \times 2^{40} \times 2^{30} =$$

$$(2^{40} \times 2^{40}) \times (2^{40} \times 2^{30}) =$$

$$2^{80} \times 2^{70} =$$

$$2^{150}$$

So, $n = 150$.

Problem 56:

I think Gelfand wants you to use actual numbers, but for $m, n > 0$...

$$(a^m)^{-n} = 1/(a^m)^n = 1/a^{mn} = a^{-mn}$$

For instance, $(a^m)^0 = 1$. (It doesn't matter what a^m is – raised to the 0, it always calculates to 1 by definition.) $1 = a^0 = a^{m \times 0}$

Problem 57:

Truthfully, we should be proving most of these statements using the Principle of Mathematical Induction, but that might be a bit much to expect, so all throughout here Gelfand is trying to get the student to (heuristically) justify each assertion to themselves. So, here we can check the formula, but how about having the child just write out the left side as ab multiplied by itself n times (using curly brackets and " n times"). Grouping the a 's and the b 's, respectively, we get a multiplied by itself n times multiplied by b multiplied by itself n times. But, just to do the exercise:

$$(a^5) \times (b^5) = aaaaabbbbb = ababababab = (ab)^5$$

$$(a^{-5}) \times (b^{-5}) = \frac{1}{aaaaa} \times \frac{1}{bbbbb} = \frac{1}{aaaaabbbbb} = \frac{1}{ababababab} = \frac{1}{(ab)^5} = (ab)^{-5}$$

Problem 58:

$$(-a)^{775} = (-1 \times a)^{775} = (-1)^{775} a^{775} = (-1)[(-1)^2]^{387} a^{775} = (-1)1^{387} a^{775} = (-1)a^{775} = -a^{775}$$

Problem 59:

$$\left(\frac{a}{b}\right)^n = \left[a\left(\frac{1}{b}\right)\right]^n = (ab^{-1})^n = a^n b^{-n}$$

Problem 60:

We would like it to be the case that $(a^p)^q = a^{pq}$ where p and q can be any number of the form n/m for integers n and m . This formula is similar to the case for integers. Then, in particular, for a given integer n , we would like $(a^{1/n})^n = a^{(1/n)n} = a^1 = a$. Or, in other words, $a^{1/n}$ is that number such that when raised to the n^{th} power is equal to a . So, $4^{1/2} = 2$ because $2^2 = 4$, and $27^{1/3} = 3$ because $3^3 = 27$.

Problem 61:

- a) $101^2 = (100 + 1)^2 = 100^2 + 2 \times 100 \times 1 + 1^2 = 10,000 + 200 + 1 = 10,201$
- b) $1,002^2 = (1,000 + 2)^2 = 1,000^2 + 2 \times 1,000 \times 2 + 2^2 = 1,000,000 + 4,000 + 4 = 1,004,004$

Problem 62:

(This problem and the one above it may well be review if your child learned arithmetic in Singapore.) Suppose I have a product of two numbers a and b . If both a and b become 10% bigger, then the new product is $(a + 10\% \text{ of } a) \times (b + 10\% \text{ of } b) = (a + 0.1a) \times (b + 0.1b) = (1.1a)(1.1b) = 1.1^2 ab = (1 + 2 \times 1 \times (0.1) + (0.1)^2) \times ab = (1 + 0.2 + 0.01) \times ab = 1.21ab$. So, the product is just over 20% bigger.

Problem 63:

Well, I am not entirely sure what this problem is getting at. Please someone (quickly) clue me in (before too many people read my answer) if necessary. A son can only have one father but a father can have many sons. So, the father of a son is the man that begat the son, so that if x is NN 's son, then x 's father must be none other than NN . On the other hand, a father can have more than one son, so that NN 's father's son could be someone else besides NN .

Problem 65:

- a) $99^2 = (100 - 1)^2 = 100^2 - 2 \times 100 \times 1 + (-1)^2 = (100 - 2) \times 100 + 1 = 9,800 + 1 = 9,801$
- b) $998^2 = (1,000 - 2)^2 = 1,000^2 - 2 \times 1,000 \times 2 + (-2)^2 = (1,000 - 4) \times 1,000 + 4 = 996,004$

Problem 66:

- a) $a = b$
- i. $4b^2 = 2^2 b^2 = (2b)^2 = (b + b)^2 = b^2 + 2 \times b \times b + b^2 = b^2 + 2b^2 + b^2 = (1 + 2 + 1)b^2 = 4b^2$

$$\begin{aligned} \text{ii. } 0 &= 0^2 = (\mathbf{b} - \mathbf{b})^2 = \mathbf{b}^2 - 2 \times \mathbf{b} \times \mathbf{b} + \mathbf{b}^2 = \mathbf{b}^2 - 2\mathbf{b}^2 + \mathbf{b}^2 = \\ &= (1 - 2 + 1)\mathbf{b}^2 = 0^2 = 0 \end{aligned}$$

$$\text{b) } a = 2b$$

$$\text{i. } 9b^2 = 3^2b^2 = (3b)^2 = (\mathbf{2b} + \mathbf{b})^2 = (\mathbf{2b})^2 + 2 \times \mathbf{2b} \times \mathbf{b} + \mathbf{b}^2 = 2^2b^2 + 4b^2 + b^2 = 4b^2 + 4b^2 + b^2 = (4 + 4 + 1)b^2 = 9b^2$$

$$\text{ii. } b^2 = (\mathbf{2b} - \mathbf{b})^2 = (\mathbf{2b})^2 - 2 \times \mathbf{2b} \times \mathbf{b} + \mathbf{b}^2 = 2^2b^2 - 4b^2 + b^2 = 4b^2 - 4b^2 + b^2 = (4 - 4 + 1)b^2 = b^2$$

Problem 68:

$$101 \times 99 = (100 + 1) \times (100 - 1) = 100^2 - 1^2 = 10,000 - 1 = 9,999$$

Problem 69:

You would cut the "top" rectangle that is b by $(a - b)$ off and rotate it 90 degrees and put it next to the figure resulting from cutting it off. Now you have a figure that is $a - b$ tall and $a + b$ wide. We can readily see from this exercise that $a^2 - b^2 = (a - b)(a + b)$ based on the areas of the two figures.

Problem 72:

Gelfand mentions "some n ", so using that notation, any given number ending in 5 that we want to square can be represented as $10n + 5$. Squaring we get $(10n + 5)^2 = 100n^2 + 100n + 25 = 100(n^2 + n) + 25 = 100 \times n \times (n + 1) + 25$. Multiplying a number by 100 and adding 25 is the same as taking the decimal expression of the number and putting a 2 and a 5 on the end.

Problem 74:

From Problem #73 we have that $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$. Substitute a " $-c$ " in this equation everywhere you see a " c " and you get: $(a + b + (-c))^2 = a^2 + b^2 + (-c)^2 + 2ab + 2a(-c) + 2b(-c)$ so that we have $(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.

Problem 75:

$(a + b + c)(a + b - c) = ((a + b) + c)((a + b) - c)$. Using the formula given in Problem #67 and substituting $(a + b)$ for a and c for b , we have $((a + b) + c)((a + b) - c) = (a + b)^2 - c^2 = a^2 + 2ab + b^2 - c^2$.

Problem 76:

$(a + b + c)(a - b - c) = (a + (b + c))(a - (b + c))$. Using the formula given in Problem #67 and substituting $(b + c)$ for b , we have $(a + (b + c))(a - (b + c)) = a^2 - (b + c)^2 = a^2 - (b^2 + 2bc + c^2) = a^2 - b^2 - 2bc - c^2$.

Problem 77:

$(a+b-c)(a-b+c) = (a+(b-c))(a-(b-c))$. Using the formula given in Problem # 67 and substituting $(b-c)$ for b , we have $(a+(b-c))(a-(b-c)) = a^2 - (b-c)^2 = a^2 - (b^2 - 2bc + c^2) = a^2 - b^2 + 2bc - c^2$.

[Top](#)

Problem 79:

$11^3 = (a+b)^3$ where $a = 10$ and $b = 1$. Then, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ by the formula in the book. Substituting 10 and 1 in the formula is easy because using $b=1$ just causes the b to drop out leaving a 1 behind and 10 to successive powers corresponds to places in the decimal form of the number. So, $11^3 = 1 \times 10^3 + 3 \times 10^2 + 3 \times 10^1 + 1 = 1,331$. (In other words, you can just strip off the coefficients of the terms in the expansion of $(a+b)^3$ and string them together to get the decimal representation of 11^3 .)

[Top](#)

Problem 80:

This time we will be taking powers of 100 instead of ten since $101^3 = (100 + 1)^3$. So, you will just string the coefficients of the expansion of $(a+b)^3$ together with zeros in between like so: 1030301. To check this answer observe that $(100 + 1)^3 = 100^3 + 3 \times 100^2 + 3 \times 100 + 1 = 1,000,000 + 30,000 + 300 + 1 = 1,030,301$.

[Top](#)

Problem 82:

We saw in problem # 79 that 11^3 could be determined just by stringing the numerical coefficients of $(a+b)^3$ together because $11 = a + b$ where $a = 10$ and $b = 1$. When we expand $(10 + 1)^n$ out we get successive powers of 10 each multiplied by a coefficient of some sort. Such an expression is just the very meaning of a decimal representation (i.e. 123,456 *means* $1 \times 10^5 + 2 \times 10^4 + 3 \times 10^3 + 4 \times 10^2 + 5 \times 10^1 + 6 \times 10^0$). So to calculate 11 raised to successive powers, n , just strip the coefficients from the binomial expansion of $(x+1)^n$ and slap them together (write them down in order) to get the decimal representation of 11^n . Using Pascal's Triangle in the book, then, we see that: $11^3 = 1,331$, and $11^4 = 14,641$.

The next row is a bit tricky since it has coefficients greater than 10, so let's write it out a bit more:

$$\begin{array}{rcl}
 11^5 & & = \\
 (10+1)^5 & & =
 \end{array}$$

$$\begin{array}{rclclclclclclclclclclclclclcl}
1 \times 10^5 & + & 5 \times 10^4 & + & 10 \times 10^3 & + & 10 \times 10^2 & + & 5 \times 10 & + & 1 & = \\
1 \times 10^5 & + & 5 \times 10^4 & + & 10^4 & + & 10^3 & + & 5 \times 10 & + & 1 & = \\
1 \times 10^5 & + & (5+1) \times 10^4 & + & 10^3 & + & 5 \times 10 & + & 1 & = \\
1 \times 10^5 & + & 6 \times 10^4 & + & 1 \times 10^3 & + & 0 \times 10^2 & + & 5 \times 10 & + & 1 & = \\
161,051
\end{array}$$

Essentially, we just write the coefficients down in the same manner as before, but "carry the one" (as in regular arithmetic) for any coefficients greater than 9. Then 11^6 will be the concatenation of 1, (6+1), (5+2), (0+1), 5, 6, 1 or in other words 1,771,561. Just in case we feel uneasy about that, let's write it out again:

$$\begin{array}{rclclclclclclclclclclclclclcl}
11^6 & = \\
(10+1)^6 & = \\
1 \times 10^6 & + & 6 \times 10^5 & + & 15 \times 10^4 & + & 20 \times 10^3 & + & 15 \times 10^2 & + & 6 \times 10 & + & 1 & = \\
1 \times 10^6 & + & 6 \times 10^5 & + & (10+5) \times 10^4 & + & 2 \times 10 \times 10^3 & + & (10+5) \times 10^2 & + & 6 \times 10 & + & 1 & = \\
1 \times 10^6 & + & 6 \times 10^5 & + & 10 \times 10^4 & + & 5 \times 10^4 & + & 2 \times 10 \times 10^3 & + & 10 \times 10^2 & + & 5 \times 10^2 & + & 6 \times 10 & + & 1 & = \\
1 \times 10^6 & + & 6 \times 10^5 & + & 10^5 & + & 5 \times 10^4 & + & 2 \times 10^4 & + & 10^3 & + & 5 \times 10^2 & + & 6 \times 10 & + & 1 & = \\
1 \times 10^6 & + & (6+1) \times 10^5 & + & (5+2) \times 10^4 & + & 10^3 & + & 5 \times 10^2 & + & 6 \times 10 & + & 1 & = \\
1 \times 10^6 & + & 7 \times 10^5 & + & 7 \times 10^4 & + & 7^4 & + & 5 \times 10 & + & 1 & = \\
1 \times 10^6 & + & 7 \times 10^5 & + & 7 \times 10^4 & + & 1 \times 10^3 & + & 5 \times 10^2 & + & 6 \times 10 & + & 1 & = \\
1,771,561
\end{array}$$

[Top](#)

Problem 83:

What we need to do is fill the question marks in on: $(a+b)^7 = ? \times a^7 b^0 + ? \times a^6 b^1 + ? \times a^5 b^2 + ? \times a^4 b^3 + ? \times a^3 b^4 + ? \times a^2 b^5 + ? \times a^1 b^6 + ? \times a^0 b^7$. The question marks come from the next row of Pascals Triangle:

```

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

```

The last row is gotten by putting a 1 on each end and then each term inbetween is the sum of the terms in the row above it. Thus, the question marks are just the numbers in the last row: $(a+b)^7 = 1 \times a^7 b^0 + 7 \times a^6 b^1 + 21 \times a^5 b^2 + 35 \times a^4 b^3 + 35 \times a^3 b^4 + 21 \times a^2 b^5 + 7 \times a^1 b^6 + 1 \times a^0 b^7$.

[Top](#)

Problem 84:

We can just substitute $-b$ in everywhere we see a b in the preceding formulas because $a-b = a+(-b)$. Thus:

$$\begin{aligned} (a-b)^4 &= \\ 1 \times a^4(-b)^0 &+ 4 \times a^3(-b)^1 + 6 \times a^2(-b)^2 + 4 \times a^1(-b)^3 + 1 \times a^0(-b)^4 = \\ a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 & \end{aligned}$$

$$\begin{aligned} (a-b)^5 &= \\ 1 \times a^5(-b)^0 &+ 5 \times a^4(-b)^1 + 10 \times a^3(-b)^2 + 10 \times a^2(-b)^3 + 5 \times a^1(-b)^4 + 5 \times a^0(-b)^5 = \\ a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 + b^5 & \end{aligned}$$

$$\begin{aligned} (a-b)^6 &= \\ 1 \times a^6(-b)^0 &+ 6 \times a^5(-b)^1 + 15 \times a^4(-b)^2 + 20 \times a^3(-b)^3 + 15 \times a^2(-b)^4 + 6 \times a^1(-b)^5 + 1 \times a^0(-b)^6 = \\ a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6 & \end{aligned}$$

[Top](#)

Problem 85:

$$\begin{array}{ll} 1 & =1 \\ 1 + 1 & =2 \\ 1 + 2 + 1 & =4 \\ 1 + 3 + 3 + 1 & =8 \\ 1 + 4 + 6 + 4 + 1 & =16 \\ 1 + 5 + 10 + 10 + 5 + 1 & =32 \\ 1 + 6 + 15 + 20 + 15 + 6 + 1 & =64 \\ 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 & =128 \end{array}$$

The sums are forming the powers of 2: $1 = 2^0$, $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, $16 = 2^4$, $32 = 2^5$, $64 = 2^6$, and $128 = 2^7$. Just following the pattern, you would expect the next sum to be equal to $2^8 = 256$. Let's do an ingenious little trick here:

$$\begin{aligned} 2^8 &= (1+1)^8 = \\ 1 \times 1^8 1^0 &+ 8 \times 1^7 1^1 + 28 \times 1^6 1^2 + 56 \times 1^5 1^3 + 70 \times 1^4 1^4 + 56 \times 1^3 1^5 + 28 \times 1^2 1^6 + 8 \times 1^1 1^7 + 1 \times 1^0 1^8 = \\ 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 & \end{aligned}$$

In other words, 2^8 is indeed just the sum of the coefficients of the 8th row of Pascal's Triangle. We can do this for any n , so the sum of the $n+1$ row in Pascal's Triangle is 2^n .

[Top](#)

Problem 86:

I am just going to write this out for

$$(a+b)^4 = 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4$$

Substituting a in for b , we get:

$$(a+a)^4 = 1a^4a^0 + 4a^3a^1 + 6a^2a^2 + 4a^1a^3 + 1a^0a^4$$

or

$$(2a)^4 = 1a^4 + 4a^4 + 6a^4 + 4a^4 + 1a^4$$

or

$$2^4a^4 = (1 + 4 + 6 + 4 + 1)a^4$$

Or, in other words, the sum of the coefficients of the $n+1$ row of Pascal's Triangle is 2^n . We already knew this from the last problem, but it is nice to watch the arithmetic work out (and maybe not all of us *did* quite see this in the last problem).

[Top](#)

Problem 87:

Let's all say it together... "The sum of the terms in the $n+1$ row of Pascal's Triangle is 2^n , amen." You can also say it "The sum of the coefficients of the binomial expansion of $(a+b)^n$ is 2^n ," but Gelfand hasn't actually used that terminology ("binomial expansion"), yet.

[Top](#)

Problem 88:

Again, I am just going to write this out for $(a+b)^4 = 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4$. Substituting $-b$ in for a , we get:

$$(-b+b)^4 = 1(-b)^4b^0 + 4(-b)^3b^1 + 6(-b)^2b^2 + 4(-b)^1b^3 + 1(-b)^0b^4$$

or

$$(0)^4 = 1b^4 - 4b^4 + 6b^4 - 4b^4 + 1b^4$$

or

$$0 = (1 - 4 + 6 - 4 + 1)b^4$$

Or, in other words, the sum of the coefficients of the $n+1$ row of Pascal's Triangle *with alternating signs* is zero! Now, that is pretty neat!

[Top](#)

Problem 90:

This problem is a lot more like 8 problems! I think that we are mainly trying to gain some facility here, so really try to get them on your own and/or have your child try to get them on their own.

(a) $(1+x)(1+x^2) = (1+x)1 + (1+x)x^2 = 1 + x + (1x^2 + xx^2) = 1 + x + x^2 + x^3$

(b) Looking at what happened in the last problem, we might be able to see that this product is just going to be the sum of powers of x up to $x^{8+(8-1)} = x^{15}$. But let's see:

$$\begin{aligned} & (1+x)(1+x^2)(1+x^4)(1+x^8) \\ &= \\ &= [(1+x+x^2+x^3)1] + [(1+x+x^2+x^3)x^4](1+x^8) \\ &= [1+x+x^2+x^3+(1x^4+xx^4+x^2x^4+x^3x^4)](1+x^8) \\ &= (1+x+x^2+x^3+x^4+x^5+x^6+x^7)(1+x^8) \\ &= (1+x+x^2+x^3+x^4+x^5+x^6+x^7)1 + (1+x+x^2+x^3+x^4+x^5+x^6+x^7)x^8 \\ &= 1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}+x^{12}+x^{13}+x^{14}+x^{15} \\ &= 1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}+x^{12}+x^{13}+x^{14}+x^{15} \end{aligned}$$

(c) Without resorting to any trickery, we will just do a straight forward calculation here:

$$\begin{aligned} & (1+x+x^2+x^3)^2 \\ &= (1+x+x^2+x^3)(1+x+x^2+x^3) \\ &= 1(1+x+x^2+x^3) + x(1+x+x^2+x^3) + x^2(1+x+x^2+x^3) + x^3(1+x+x^2+x^3) \\ &= (1+x+x^2+x^3) + (x+x^2+x^3+x^4) + (x^2+x^3+x^4+x^5) + (x^3+x^4+x^5+x^6) \end{aligned}$$

$$\begin{array}{r} 1 + x + x^2 + x^3 \\ x + x^2 + x^3 + x^4 \\ x^2 + x^3 + x^4 + x^5 \\ x^3 + x^4 + x^5 + x^6 \\ \hline \end{array}$$

$$1 + 2x + 3x^2 + 4x^3 + 3x^4 + x^5 + x^6$$

- (d) The vertical addition problem doesn't fit on the page well. I suspect that Gelfand is sort of doing this on purpose to get you to try to think through it without writing everything down or at least to come up with a good way to write everything down. Because there are 11 terms in the sum, when we distribute one sum through a copy of it, we will get the sum of 11 sums. In any given sum, the powers of x are all one greater than the sum before it. So, we end up with something like:

$$\begin{array}{cccccccccccc}
 1+ & x+ & \dots & +x^9+ & x^{10} & & & & & & & \\
 & x+ & x^2+ & \dots & +x^{10}+ & x^{11} & & & & & & \\
 & \dots & & & & & & & & & & \\
 & & & & x^9+ & x^{10}+ & \dots & +x^{18}+ & x^{19} & & & \\
 & & & & x^{10}+ & x^{11}+ & \dots & +x^{19}+ & x^{20} & & & \\
 \hline
 1+ & 2x+ & \dots & +10x^9+ & 11x^{10}+ & 10x^{11}+ & \dots & +2x^{19}+ & x^{20}
 \end{array}$$

Notice that the coefficient of x^{10} is 11 and the coefficients increase to 11 and then decrease back down to one.

- (e) First we can just substitute the above solution in for two of the terms in the product to get the above solution multiplied by the original $(1+x+\dots+x^{10})$. As we multiply this product out, we will again get the sum of 11 sums. However, now each sum will be based on what was given as the answer to the last problem rather than on the original $(1+x+\dots+x^{10})$. Each sum's powers of x will be one greater than the sum before it, and so on. Rather than write them all down, we really only need to look at the last couple to get the coefficients of x^{29} and x^{30} since 30 is the largest possible power of x in the product. In fact, for x^{30} , the only way to get such a term is by multiplying the x^{10} by the x^{20} term in the final product, so the coefficient of x^{30} must be one. The x^{29} can be obtained by multiplying the x^{10} term by the x^{19} term or by multiplying the x^9 term by the x^{20} term so we can just add the coefficients of the x^{19} term and the x^{20} term to get $2+1 = 3$ as the coefficient of the x^{29} term.
- (f) This multiplication corresponds to a very important trick we will use later. Distributing the $(1-x)$ through we get

$$\begin{aligned}
 & (1-x)(1+x+\dots+x^9+x^{10}) \\
 &= \\
 &= (1-x) + (x-x^2) + \dots + (1-x)1 + (1-x)x + \dots + (1-x)x^9 + (1-x)x^{10} \\
 &= 1 + (-x+x) + (-x^2+x^2) + \dots + (-x^9+x^9) + (-x^{10}+x^{10}) + (-x^{11}) \\
 &= 1 - x^{11}
 \end{aligned}$$

Observe that when we regroup terms, all the inner terms cancel leaving us only with the first and last term, so that $(1-x)(1+x+\dots+x^9+x^{10}) = 1-x^{11}$.

- (g) Again, I am not going to use any trickery -- just a straightforward calculation:

$$\begin{aligned}
 & (a+b)(a^2-ab+b^2) \\
 &= (a+b)a^2 - (a+b)ab + (a+b)b^2
 \end{aligned}$$

$$\begin{aligned}
&= \quad aa^2 \quad + \quad ba^2 \quad - \quad (aab \quad + \quad bab) \quad + \quad ab^2 \quad + \quad bb^2 \\
&= a^3 + ba^2 - a^2b - ab^2 + ab^2 + b^3 = a^3 + b^3
\end{aligned}$$

So, $(a+b)(a^2-ab+b^2) = a^3 + b^3$. That also means that if you ever just see $(a^3 + b^3)$, you could write as $(a+b)(a^2-ab+b^2)$ which we will find out is called factoring it.

- h. We will have to employ some trickery here. Observe the alternating terms of $1 - x + x^2 - x^3 + \dots - x^9 + x^{10}$. Group on each difference and rewrite the sum to get:

$$\begin{aligned}
1 - x + x^2 - x^3 + \dots x^9 + x^{10} &= \\
(1 - x) + (x^2 - x^3) + \dots + (x^8 - x^9) + x^{10} &= \\
(1 - x) + x^2(1 - x) + \dots + x^8(1 - x) + x^{10} &= \\
(1 - x)(1 + x^2 + \dots + x^8) + x^{10} &=
\end{aligned}$$

Now, multiply the other part of the product:

$$\begin{aligned}
&((1 - x)(1 + x^2 + \dots + x^8) + x^{10})(1 + x + x^2 + \dots + x^{10}) = \\
&(1 + x^2 + \dots + x^8)(1 - x)(1 + x + \dots + x^{10}) + x^{10}(1 + x + \dots + x^{10}) = \\
&(1 + x^2 + \dots + x^8)(1 - x^{11}) + x^{10}(1 + x + \dots + x^{10}) = \\
&1 + x^2 + \dots + x^8 - x^{11} - x^{13} - \dots - x^{19} + x^{10} + x^{11} + x^{12} + \dots + x^{20} = \\
&1 + x^2 + \dots + x^8 - x^{11} + x^{11} - \dots - x^{19} + x^{19} + x^{10} + x^{12} + \dots + x^{20} = \\
&1 + x^2 + \dots + x^8 + x^{10} + x^{12} + \dots + x^{20}
\end{aligned}$$

Did you see that step where I used part (f)? Actually, this part of the problem wasn't the important trick I was referring to above, but it sure was useful here, wasn't it?

[Top](#)

Problem 92:

Substituting -1 for x in the formula on the left side of the equation gives us zero. Substituting -1 into the formula on the right side can only give us zero is the "... " is equal to -1. Similarly, we will get zero if we substitute -3 into x in the formula on the right side of the equation, and the only way we get zero from the formula on the left side is if the "... " is -3.

[Top](#)

Problem 96:

I'm not sure if Gelfand thinks it is clear that it cannot happen and that you should reconsider the problem later if you think that it can. Or, perhaps Gelfand is saying that if you think this problem is silly, then you should reconsider it several years later. What about $(x-x)*P(x)$? All the terms cancel in that product, but the first term of the product is not really in its proper form. However, it might not be so hard to imagine how such a thing might happen given this example which isn't really an example of it actually happening like Gelfand is imagining in the question. In a more advanced class this question is

asking whether or not $\mathbf{R}[x]$ is an Integral Domain, and such a fact is not to me taken for granted. For now imagine two polynomials, $P(x)$ and $Q(x)$, that do not just reduce to zero (like in the example of " $x-x$ "). Then, you might find it more believable (though still unproven here) that there will be a number r such that $P(r)$ is not zero and $Q(r)$ is not zero. Then, the product $P(r)*Q(r)$ is not zero, so all the terms can't have cancelled out.

[Top](#)

Problem 100:

We've seen this before -- see problems 66 and 88. Well, the sum of terms in Pascal's Triangle using $a=1$ and $b=-1$ is going to clearly be zero because the sum is equal to $(1-1)$ raised to the n^{th} power or zero raised to the n^{th} power. What may not be quite as obvious is that using $a=1$ and $b=-1$ we will get alternating signs in the terms of Pascal's Triangle. The reason we get alternating signs is because b is raised sequentially to higher and higher powers starting from zero and adding one in each term all the way up to raising b to the n^{th} power in the last term. On the even powers (including zero), we will get that (-1) raised to such a power just equals 1 and we get the corresponding term in Pascal's Triangle adds to the sum. For odd powers, we get that (-1) raised to such a power is equal to -1 and so the corresponding term in Pascal's Triangle, being multiplied by this negative 1, subtracts from the sum. Thus we get the alternating signs because the powers of b alternate between being even and odd (zero is even, 1 is odd, 2 is even, 3 is odd, and so on).

[Top](#)

Problem 101:

Just set $x=1$ in the expansion and you will get the sum of coefficients. Then such a sum must equal $(1+2 \times 1)^{200} = 3^{200}$. I doubt you are actually supposed to calculate that out.

[Top](#)

Problem 102:

Just set $x=-1$ again in the expansion and you will get the sum of coefficients. Then such a sum must equal $(1-2 \times 1)^{200} = (-1)^{200} = 1$.

[Top](#)

Problem 103:

Just as in the previous two problems, set $x=1$ and $y=1$ to get $(1+1-1)^3 = 1^3 = 1$.

[Top](#)

Problem 104:

Set $x=1$ and $y=0$ to get all the terms not including y . Thus, we get $(1+1-0)^3=8$.

[Top](#)

Problem 105:

The sum of the terms containing x is the sum of the terms not containing y less the sum of the constant terms. There can be only one constant term which is the result of cubing the 1 in $(1+x-y)^3$. Then the sum of the coefficients of the terms containing x (based on the previous problem) must be $8-1=7$.

[Top](#)

Problem 107:

$$(a) \quad ac+bc-ad-bd \quad = \quad c(a+b)-d(a+b) \quad = \quad (a+b)(c-d)$$

(b) $1+a+a^2+a^3 = (1+a)+a^2(1+a) = (1+a)(1+a^2)$. And, $1+a^2$ cannot actually be factored because it does not have real roots.

(c)

- i) First of all I wonder if there is an error here in having this series not sum to a^{15} . Had he done that, we could have just iteratively applied the trick in part (b) to get

$$\begin{aligned} 1+a+\dots+a^{15} &= (1+a)+(a^2)(1+a)+\dots+(a^{14})(1+a) \\ &= (1+a)(1+a^2+a^4+\dots+a^{14}) \\ &= (1+a)(1+a^2)+a^4(1+a^2)+\dots+a^{12}(1+a^2) \\ &= (1+a)(1+a^2)(1+a^4+a^8+a^{12}) \\ &= (1+a)(1+a^2)[(1+a^4)+(a^8)(1+a^4)] \\ &= (1+a)(1+a^2)(1+a^4)(1+a^8) \end{aligned}$$

As long as the number of terms in the sum are equal to some power of two (in this case, we had $16 = 2^4$ terms), you can always do this trick. And then secondly, I think that Gelfand's

idea of factoring might be just to factor out one thing or so as opposed to really reduce it to a product of irreducible factors. (So we would leave this factored as above even though $(1+a^4)$ and $(1+a^8)$ can both be further factored.)

- ii) With that said, let me try and tackle this problem as it actually appears in the book. If you are paying careful attention, you will see that the above trick won't work on this progression because it has an *odd* number of terms. So try it in groups of three which does evenly divide the number (15) of terms:

$$\begin{aligned} & 1+a+a^2+a^3+a^4+a^5+a^6+a^7+a^8+a^9+a^{10}+a^{11}+a^{12}+a^{13}+a^{14} \\ &= (1+a+a^2)+a^3(1+a+a^2)+a^6(1+a+a^2)+a^9(1+a+a^2)+a^{12}(1+a+a^2) \\ &= (1+a+a^2)(1+a^3+a^6+a^9+a^{12}) \end{aligned}$$

Now as it turns out, the second progression can indeed be further factored; however, I have no idea how a student at this point would figure out what the factors are. Let $c = (1+\sqrt{5})/2$ and $d = (1-\sqrt{5})/2$. Then, $1+a^3+a^6+a^9+a^{12} = (1+ca^3+a^6)(1+da^3+a^6)$. And, I believe each of these can be further factored into a product of three quadratics. For instance, $1+ca^3+a^6 = (1+pa+a^2)(1+qa+a^2)(1+ra+a^2)$ where $r=1+\sqrt{c}$, $q=[-r+\sqrt{(5r^2-8)}]/2$, and $p=[-r+\sqrt{(5r^2-8)}]/2$. A similar formula using d instead of c would be used for the other factor. (Although, I haven't rigorously checked these.) And, it cannot be factored beyond that because the sum has no real roots.

To see that it has no real roots, recall that in one of the parts to problem 90, we found that $(1-x)(1+x+\dots+x^{10})=1-x^{11}$. Well, here again we would get a similar result that $(1-a)(1+a+\dots+a^{14})=1-a^{15}$, or in other words, our sum $1+a+\dots+a^{14}$ equals $(1-a^{15})/(1-a)$ so long as a is not equal to 1. But, whenever $1-a$ is negative, so is $1-a^{15}$ which shows that for all a not equal to 1 our sum $(1+a+\dots+a^{14})$ is always greater than zero since $(1-a^{15})/(1-a)$ could only equal zero at $a=1$. And plugging 1 into a in our sum shows that the sum is still positive (equal to 15, as a matter of fact). And so, the point is that our sum is always positive which shows that it cannot have any roots (which is just a point at which the sum equals zero). But, if it cannot have any roots, then it cannot ever have a factor of the form $(ra+s)$ for some constants r and s since such a factor would mean that plugging $(-s/r)$ in for a into our sum would cause it to add to zero.

$$(d) \ x^4-x^3+2x-2 = x^3(x-1)+2(x-1) = (x^3+2)(x-1)$$

[Top](#)

Problem 109:

$$(a) \ a^2-3ab+2b^2 = a^2-2ab-ab+2b^2 = a(a-2b)-(ab-2b^2) = a(a-2b)-b(a-2b) = (a-b)(a-2b)$$

$$(b) \ a^2+3a+2 = a^2+2a+a+2 = a(a+2)+(a+2) = (a+1)(a+2)$$

[Top](#)

Problem 110:

- (a) $a^2+4ab+4b^2 = a^2+2a(2b)+(2b)^2$. You may recognize this as a perfect square using $2b$ rather than just b , so that our factorization is $(a+2b)^2$.
- (b) $a^4+2a^2b^2+b^4$ is just a perfect square using a *squared* instead of a and b squared instead of b . Thus our factorization is $a^2+(b^2)^2$.
- (c) $a^2-2a+1 = a^2+2a(-1)+(-1)^2$ is just a perfect square using $b=-1$. So, our factorization is $(a-1)^2$.

[Top](#)

Problem 112:

I will (from now on) use the symbol " $\Rightarrow$ " to mean "implies". That is, "if p then q " is expressed symbolically as " $p \Rightarrow q$ ". And more concretely, if it is always true that "if Socrates is a man then Socrates is mortal," then we can write "Socrates is a man" $\Rightarrow$ "Socrates is mortal". On the one hand, it doesn't matter if Socrates is, indeed, a man for the " $\Rightarrow$ " to still hold, but on the other hand, if there is ever a case where a man could not be mortal *even if Socrates does happen to be mortal* then the " $\Rightarrow$ " cannot be used.

$$a^2=b^2 \Rightarrow a^2-b^2=0 \Rightarrow (a-b)(a+b)=0 \Rightarrow a-b=0 \text{ or } a+b=0 \Rightarrow a=b \text{ or } a=-b.$$

[Top](#)

Problem 116:

- (a) Using the same sort of tricks we employed in previous problems:

$$\begin{aligned}
 &a^5-b^5 \\
 &= a^5-a^4b+a^4b-a^3b^2+a^3b^2-a^2b^3+a^2b^3-ab^4+ab^4-b^5 \\
 &= a^4(a-b)+a^3b(a-b)+a^2b^2(a-b)+ab^3(a-b)+b^4(a-b) \\
 &= (a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)
 \end{aligned}$$

I'm not sure that we aren't supposed to stop here, but it does turn out that $a^4+a^3b+a^2b^2+ab^3+b^4$ can be further factored into the product of two quadratics:

$$a^4+a^3b+a^2b^2+ab^3+b^4 = (a^2+cab+b^2)(a^2+dab+b^2)$$

where $c=(1+\sqrt{5})/2$ and $d=(1-\sqrt{5})/2$.

Figuring this out is not easy (and I did it for 107(c) which I thought must have been a mistake as you may recall), but I will tell you here how I did it. First of all, we know that this polynomial can be expressed as the product of two quadratics because it is a fourth degree polynomial and the fact that we are always able to factor a polynomial with real coefficients into quadratics. (I am not going to say why the latter is true – it really is beyond the scope of this text.) So just try to think of a couple of quadratics. They will be of the form $Ra^2+cab+Sb^2$ for some constants R , c and S . Since the coefficients of a^4 and b^4 have to be 1 and the only way to get these coefficients is in the one product of the a^2 terms in each quadratic and the b^2 from each, respectively, we know that R and S must both be 1. (Actually, you could use reciprocal fractions which multiply to one, but that will only complicate the problem – just try using one for R and S . Maybe a better way of saying it is that you do not *have* to use anything other than one for R and S .) Thus, we have can multiply out $(a^2+cab+b^2)(a^2+dab+b^2)$ and collect like terms to eventually get: $a^4+(c+d)a^3b+(cd+2)a^2b^2+(c+d)ab^3+b^4$.

But the coefficients are all supposed to be 1 so that $c+d=1$ and $cd+2=1$. We can solve for the values of c and d by observing from the first equation that $c=1-d$ and substituting this in for the second equation to get $(1-d)d+2=1$ or $d-d^2=-1$ or that $d^2-d-1=0$. (You are not supposed to know this trick yet, I think.) I used the quadratic formula to find values for d which you are definitely not supposed to know, but I suppose one could imagine that Gelfand is trying to elicit the student to complete the square or something. Nevertheless, using the formula, we get $d=[(-1)-\sqrt{(-1)^2-4(1)(-1)}}/2=(1-\sqrt{5})/2$. Actually, it could also equal $(1+\sqrt{5})/2$, according to the quadratic formula, but subtracting either one from 1 yields the other. So, whichever one d is, c is the other. And, that gives us our unknown constants in our factorization. And, sure enough, if you multiply it out, you get the polynomial we are trying to factor. (I don't think Gelfand really expected the student to figure all this out which is why I think you were supposed to stop at factoring out the $a-b$. You can also show using the quadratic formula that the quadratics cannot be further factored using real numbers as coefficients, so this is the complete factorization.)

- (b) Using the difference of squares formula: $a^{10}-b^{10} = (a^5)^2-(b^5)^2 = (a^5-b^5)(a^5+b^5)$. Perhaps we are supposed to stop here, but we did already factor (a^5-b^5) in part (a) so we could just write that factorization down in place of (a^5-b^5) . Furthermore, substituting $-b$ in for a in (a^5+b^5) , we get $(-b)^5+b^5 = -b^5+b^5 = 0$. But, that can only happen if $a-(-b)=(a+b)$ is a factor of (a^5+b^5) . We need to multiply $(a+b)$ by some quartic polynomial (4th degree polynomial) so that the cross terms all cancel each other out. Try using a quartic with alternating signs such as $a^4-a^3b+a^2b^2-ab^3+b^4$. Multiplying by $(a+b)$, we get

$$\begin{aligned} & (a+b)(a^4-a^3b+a^2b^2-ab^3+b^4) \\ &= \\ &= \\ &= \\ &= a^5+b^5 \end{aligned} \qquad \begin{aligned} & a^4(a+b)-a^3b(a+b)+a^2b^2(a+b)-ab^3(a+b)+b^4(a+b) \\ & (a^5+a^4b)-(a^4b+a^3b^2)+(a^3b^2+a^2b^3)-(a^2b^3+ab^4)+ab^4+b^5 \\ & a^5+(a^4b-a^4b)+(-a^3b^2+a^3b^2)+(a^2b^3-a^2b^3)+(-ab^4+ab^4)+b^5 \end{aligned}$$

So, that does, indeed, work, and we have that a^5+b^5 factors into $(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$. We can use a similar trick to what we did above in part (a) (which you are probably not expected to do) to further factor $a^4-a^3b+a^2b^2-ab^3+b^4$ into the product of two quadratics: $a^4-a^3b+a^2b^2-ab^3+b^4 = (a^2+cab+b^2)(a^2+dab+b^2)$ where $c=(-1+\sqrt{5})/2$ and $d=(-1-\sqrt{5})/2$. (Notice the negative ones in c and d as compared to the positive ones from above.) And, finally, we can see for the same reason as

above (using the discriminant from the quadratic formula) that these quadratics are unfactorable.

- (c) Plugging 1 in for a , we see that $1^7 - 1 = 0$, so $(a-1)$ must be a factor. In fact, recall some of the previous problems where we multiplied $(1-x)$, for instance, by a sum of progressively increasing powers of x . We should try such a combination here and see what happens:

$$\begin{aligned}
 &(a-1)(a^6+a^5+a^4+a^3+a^2+a+1) \\
 &= (a-1)a^6+(a-1)a^5+(a-1)a^4+(a-1)a^3+(a-1)a^2+(a-1)a+(a-1)1 \\
 &= a^7+(-a^6+a^6)+(-a^5+a^5)+(-a^4+a^4)+(-a^3+a^3)+(-a^2+a^2)+(-a+a)-1 \\
 &= a^7-1
 \end{aligned}$$

After discussing these problems with my wife, I think you are almost surely supposed to factor out one or two factors and stop there. So, in this case, the solution is

$$a^7-1 = (a-1)(a^6+a^5+a^4+a^3+a^2+a+1)$$

You can further factor $a^6+a^5+a^4+a^3+a^2+a+1$, though, using techniques similar to what I have done above. You know, for instance, that it has no real roots and so must therefore be a product of three quadratics. You can write down some simple quadratics with variables used for the middle coefficients and then try to solve for the variables. In order to do so, you will find that you have to depress a cubic, use Vieta's substitution to turn it into a quadratic and then solve it. In other words, you will have to use techniques and theorems clearly beyond the student's knowledge at this point, and I have no desire to work that hard at providing the additional factorization. (So, I'm stopping here.)

[Top](#)

Problem 118:

(a) We have to use the trickery in the preceding discussion in the book and think of 2 as the square root of 2 squared. Then, we get the difference of two squares $(a^2)-2 = (a^2)-[\sqrt{2}]^2 = [a-\sqrt{2}][a+\sqrt{2}]$.

(b) Again, try to write $3b^2$ as a perfect square: $3b^2 = [b\sqrt{3}]^2$. We use the difference of squares formula to obtain $(a^2)-3(b^2) = (a^2)-[b\sqrt{3}]^2 = [a-b\sqrt{3}][a+b\sqrt{3}]$

(c) The trick to this problem is to be able to notice that $(a^2)+2ab+(b^2) = (a+b)^2$ is a perfect square. So, we have the difference of squares again:

$$\begin{aligned}
 &(a^2)+2ab+(b^2)-(c^2) \\
 &= [(a+b)^2]-(c^2) \\
 &= [(a+b)-c][(a+b)+c]
 \end{aligned}$$

(d) No here, I am pretty sure the student is supposed to guess at completing the square. What perfect

square have we seen that has $(a^2)+4ab$ in it? It is $(a+2b)^2 = (a^2)+4ab+4(b^2)$. What we have is very close to that if we add and subtract b^2 , we'll once again get back to the difference of squares:

$$\begin{aligned}
 &(a^2)+4ab+3(b^2) \\
 = & \\
 = & \qquad \qquad \qquad (a^2)+4ab+3(b^2)+(b^2)-(b^2) \\
 = & \qquad \qquad \qquad (a^2)+4ab+4(b^2)-(b^2) \\
 = & \qquad \qquad \qquad (a+2b)^2-(b^2) \\
 = & \qquad \qquad \qquad [(a+2b)-b][(a+2b)+b] \\
 = & (a+b)(a+3b)
 \end{aligned}$$

[Top](#) [Education Main Page](#)

Problem 120:

If n is even, we can just use the same tactic as seen in problem 119 because n can be written as $2m$ for some m :

$$\begin{aligned}
 &[a^{(2n)}] \qquad \qquad \qquad + \qquad \qquad \qquad [b^{(2n)}] \\
 = & \qquad \qquad \qquad [(a^n)^2] \qquad \qquad \qquad + \qquad \qquad \qquad [(b^n)^2] \\
 = & \qquad [(a^n)^2] \qquad + \qquad 2(a^n)(b^n) \qquad + \qquad [(b^n)^2] \qquad - \qquad 2(a^n)(b^n) \\
 = & \qquad \qquad \{[(a^{2m})+(b^{2m})]^2\} \qquad \qquad \qquad - \qquad \qquad \qquad 2(a^{2m})(b^{2m}) \\
 = & \qquad \{[(a^{2m})+(b^{2m})]^2\} \qquad \qquad \qquad - \qquad \qquad \{[\sqrt{2}*(a^m)(b^m)]^2\} \\
 = & \qquad [(a^{2m})+(b^{2m})-\sqrt{2}*(a^m)*(b^m)][(a^{2m})+(b^{2m})+\sqrt{2}*(a^m)*(b^m)]
 \end{aligned}$$

If n is odd and greater than 1, then using the hint and the discussion immediately preceding this problem in the text:

$$\begin{aligned}
 &[a^{(2n)}] \qquad \qquad \qquad + \qquad \qquad \qquad [b^{(2n)}] \\
 = & \qquad \qquad \qquad [(a^2)^n] \qquad \qquad \qquad + \qquad \qquad \qquad [(b^2)^n] \\
 = & [(a^2)+(b^2)][(a^2)^{(n-1)} - [(a^2)^{(n-2)}]*(b^2) + [(a^2)^{(n-3)}]*[(b^2)^2] - \dots - (a^2)*[(b^2)^{(n-2)}] + [(b^2)^{(n-1)}]]
 \end{aligned}$$

I am writing the original polynomial as the sum of two variables raised to the same odd power. Then, whatever expressions are raised to the odd power when you are done getting the original polynomial into this form, you can factor out the sum of the two expressions. And, the remaining factor is this sum of terms with alternating signs. (That's why it only works with odd powers -- because the trick of alternating the sign to get the cross terms to cancel when you multiply the factors out won't work if there are an even number of terms.)

So, in short, yes -- we can always factor something out of such a polynomial if n is greater than 1.

[Top](#) [Education Main Page](#)

Problem 121:

Just multiply it out using the difference of squares formula:

$$\begin{aligned}
 &(2+3\sqrt{-1})(2-3\sqrt{-1}) \\
 &= \\
 &= \\
 &= 4 - 9(-1) \\
 &= 4 + 9 \\
 &= 13
 \end{aligned}$$

[Top](#) [Education Main Page](#)

Problem 122:

(a) Use the solution to Problem 119 only with x as a and 1 as b :

$$\begin{aligned}
 &(x^4+1) \\
 &= [(x^2)+1+x\sqrt{2}][(x^2)+1-x\sqrt{2}]
 \end{aligned}$$

b. Let's try grouping on one of the variables: x .

$$\begin{aligned}
 &x(y^2 - z^2) + y(z^2 - x^2) + z(x^2 - y^2) = \\
 &(z - y)x^2 + (y^2 - z^2)x + yz^2 - zy^2 = \\
 &(z - y)x^2 - (z - y)(y + z)x + yz(z - y) = \\
 &(z - y)(x^2 - (y + z)x + yz) = \\
 &(z - y)(x - y)(x - z)
 \end{aligned}$$

(c) Use the same approach as in problem # 111 to try and factor out an (a^2+a+1) . The student will probably come to try this in their last ditch effort to solve the problem after nothing else has worked. Or, they might end up doing what I did (since I didn't actually do # 111) and multiply $[(a^2)+ca+1][(a^8)+d(a^7)+e(a^6)+...+ja+1]$ out and try some nice values for c, d, e, f, g, h, i and j .

a¹⁰	+a⁵	+1 =
a¹⁰	+a ⁹ +a ⁸	
	-a ⁹ -a ⁸	-a ⁷
		+a ⁷
		+a ⁶
		-a ⁶
		+a ⁵
		-a ⁵ -a ⁴
		+a ⁵ +a ⁴ +a ³
		-a ³ -a ² -a
		+a ² +a +1 =

$$\begin{aligned}
& (a^8)*(a^2+a+1) \\
& - (a^7)*(a^2+a+1) \\
& + (a^5)*(a^2+a+1) \\
& - (a^4)*(a^2+a+1) \\
& + (a^3)*(a^2+a+1) \\
& - a*(a^2+a+1) \\
& + 1*(a^2+a+1) =
\end{aligned}$$

$$[(a^8) - (a^7) + (a^5) - (a^4) + (a^3) - a + 1] * [(a^2) + a + 1]$$

And, this is probably as far as you are supposed to factor it, especially based on the fact that Gelfand left the cubic unfactored in Problem # 111.

(d) Holy cow! This one was a toughie! The only reason I eventually guessed the right answer was because I started to look ahead and saw Problem #125 which was a huge hint that $(a+b+c)$ was probably a factor. At that point I just tried to think of how I could get cross terms to cancel in a product between $(a+b+c)$ and something that would have to have an $[(a^2)+(b^2)+(c^2)]$ in it (in order to get an $[(a^3)+(b^3)+(c^3)]$). Well if you multiply the two, you get:

$$\begin{aligned}
& (a+b+c)*[(a^2)+(b^2)+(c^2)] \\
& = (a^3)+(b^3)+(c^3)+(a^2)b+(a^2)c+a(b^2)+(b^2)c+a(c^2)+b(c^2)
\end{aligned}$$

We need to find some way of getting all those different combinations of a letter times another letter squared to cancel off. Well all those combinations are, in fact, *all* the different ways to write $x(y^2)$ (x not equal to y) with three letters a , b , and c which happens that way because of how the distributive law works. So let's try subtracting off every combination of xy (x not equal to y) with three letters so that as we multiply our $(a+b+c)$ through we will end up *subtracting off* each combination of $x(y^2)$ (x not equal to y) on three letters (a , b , and c). So it turns out that

$$\begin{aligned}
& (a+b+c)*[(a^2)+(b^2)+(c^2)-\mathbf{ab-bc-ac}] \\
& = (a^3)+(b^3)+(c^3)+(a^2)b+(a^2)c+a(b^2)+(b^2)c+a(c^2)+b(c^2)-\mathbf{(a^2)b-abc-(a^2)c-a(b^2)-} \\
& \quad \mathbf{(b^2)c-abc-abc-b(c^2)-a(c^2)}
\end{aligned}$$

And so, it turns out that all the emboldened terms cancel with the cross terms from the first product leaving 3 abc 's left over to give us our factorization:

$$\begin{aligned}
& (a+b+c)*[(a^2)+(b^2)+(c^2)-\mathbf{ab-bc-ac}] \\
& = (a^3)+(b^3)+(c^3)-3abc
\end{aligned}$$

If your child (or you) did not get this one, do not be too terribly disappointed. But, definitely tell

them the answer so they can do some of the problems that follow....

(e) First multiply out $(a+b+c)^3$ to get:

$$(a^3)+(b^3)+(c^3)+3(a^2)b+3(a^2)c+3a(b^2)+3(b^2)c+3a(c^2)+3b(c^2)+6abc$$

We should be able to readily see that if we subtract off (a^3) , (b^3) and (c^3) we will be left with:

$$3(a^2)b+3(a^2)c+3a(b^2)+3(b^2)c+3a(c^2)+3b(c^2)+6abc \\ = 3[(a^2)b+(a^2)c+a(b^2)+(b^2)c+a(c^2)+b(c^2)+2abc]$$

Perhaps if we stare at this long enough or are lucky enough to try the right thing first we might observe that:

$$(a+b)(b+c)(c+a) \\ = (a^2)b+(a^2)c+a(b^2)+(b^2)c+a(c^2)+b(c^2)+2abc$$

(Trying something like this might be a natural choice given what we have been trying to factor here in this problem.) Thus, we have, then, that

$$[(a+b+c)^3]-(a^3)-(b^3)-(c^3) \\ = 3[(a^2)b+(a^2)c+a(b^2)+(b^2)c+a(c^2)+b(c^2)+2abc] \\ = 3(a+b)(b+c)(c+a)$$

(f) Use Pascal's Triangle to multiply each cube out and then simplify:

$$(a-b)^3 = (a^3) - 3(a^2)b + 3a(b^2) - (b^3) \\ + (b-c)^3 = + (b^3) - 3(b^2)c + 3b(c^2) - (c^3) \\ + (c-a)^3 = + (c^3) - 3(c^2)a + 3c(a^2) - (a^3)$$

$$- 3(a^2)b + 3a(b^2) - 3(b^2)c + 3b(c^2) - 3(c^2)a + 3c(a^2)$$

Now, factor out a 3 and use a similar approach that we used in part (b) by grouping on one of the variables -- a.

$$-3(a^2)b+3a(b^2)-3(b^2)c+3b(c^2)-3(c^2)a+3c(a^2) \\ = 3[c(a^2)-(a^2)b+a(b^2)-(b^2)c+c(a^2)-a(b^2)+b(c^2)] \\ = 3\{(c-b)(a^2)-[(c^2)-(b^2)]*a+[b(c^2)-(b^2)c]\} \\ = 3[(c-b)(a^2)-(c-b)*(c+b)*a+bc(c-b)] \\ = 3(b-c)[(a^2)-(c+b)a+bc]$$

A slicker way of doing this that more heavily relies on some things we have done previously is to use the

sum of cubes formula in Problem # 114 (and I am going to do it so that it factors out the same way I factored above) to get:

$$\begin{aligned}
 & (a-b)^3 + (b-c)^3 + (c-a)^3 \\
 = & [(a-b)^3 + (c-a)^3] + (b-c)^3 \\
 = & [(a-b)^2 - (a-b)(c-a) + (c-a)^2](a-b+c-a) + (b-c)^3 \\
 = & (b-c)^3 - (b-c)[(a-b)^2 - (a-b)(c-a) + (c-a)^2] \\
 = & (b-c)\{(b-c)^2 - [(a-b)^2 - (a-b)(c-a) + (c-a)^2]\}
 \end{aligned}$$

And then, of course, we could multiply out and simplify the right hand term of this last expression.

[Top](#) [Education Main Page](#)

Problem 123:

The ab term is like the squared term in a univariate (aka "normal") polynomial. In fact, if (for instance) we substitute x in for both a and b we get $(1+ab)-(a+b) = 1+(x^2)-2x = (x^2)-2x+1 = (x-1)(x-1)$, so perhaps that helps us imagine that the expression will factor into $(a-1)(b-1)$. In any case:

$$\begin{aligned}
 & (1+ab)-(a+b) \\
 = & \hspace{15em} (ab-a)+1-b \\
 = & \hspace{15em} a(b-1)-(b-1) \\
 = & \hspace{15em} (a-1)(b-1)
 \end{aligned}$$

If both a and b are greater than 1, then $(a-1)$ and $(b-1)$ are each greater than zero which implies that their product is also greater than zero. That is,

$$(1+ab)-(a+b) = (a-1)(b-1) > 0$$

or

$$1+ab > a+b$$

[Top](#) [Education Main Page](#)

Problem 124:

Recalling that $[(a^3)-(b^3)]=(a-b)[(a^2)+ab+(b^2)]$, if $(a^2)+ab+(b^2)=0$, then $(a^3)-(b^3)=0$ or $(a^3)=(b^3)$ which can only happen when $a=b$. (I believe that last conclusion actually begs the question, so I will comment on that in a moment.) But if $a=b$, then $0=(a^2)+ab+(b^2)=3(b^2)$ which implies that $b=0$, and so $a=b=0$.

While I think something like this is probably what Gelfand might have had in mind, the way I have done

it begs the question. (Or, in other words, it assumes the conclusion it is supposed to prove since $(a^3)=(b^3)$ only implies that a must be equal to b because we can factor it and analyze the $(a^2)+ab+(b^2)$ term to see that such an implication holds.) So, let's analyze this term, $(a^2)+ab+(b^2)$, directly. The key here is to observe that as long as at least one of a or b is not zero, then $(a^2)+ab+(b^2)$ is always greater than zero. (We will find this out in a different way later by looking at the discriminant as Gelfand alludes to.) First note that if one of a or b is zero our equation reduces to $(a^2)+ab+(b^2) = (a^2)$ or $(a^2)+ab+(b^2) = (b^2)$, and in either case, is not equal to zero (and, being some number squared, is actually greater than zero) unless the other number is zero. To prove the general case we consider the two perfect squares $(a+b)^2$ and $(a-b)^2$ which as such are always greater than or equal to zero no matter what values we put in for a or b . Furthermore, $(a+b)^2=(a^2)+2ab+(b^2)$ while $(a-b)^2=(a^2)-2ab+(b^2)$. This last fact will allow us to compare them to the expression we are actually interested in: $(a^2)+ab+(b^2)$.

We do this in two steps -- first supposing that $a*b$ is positive and then supposing $a*b$ is negative. So, suppose $a*b$ is positive. Then, $(a^2)+ab+(b^2) > (a^2)-2ab+(b^2) = (a-b)^2 > 0$ since in such a case $a*b > 0 > -2ab$. Suppose $a*b$ is negative. Then, $(a^2)+ab+(b^2) > (a^2)+2ab+(b^2) = (a+b)^2 > 0$ since $2ab$ is "more negative" than ab in this case. (Or, more precisely: $0 > ab$ implies that $ab = 0 + ab > ab + ab = 2ab$.) So, we have seen in every case that $(a^2)+ab+(b^2)$ is always greater than zero (and in particular not equal to zero) unless $a=b=0$. (Actually, if you look closely, you will note that, for instance, $(a-b)^2$ is not always *strictly* greater than zero. It is actually *equal to zero* when $a=b$. But, in that case we get something like $3(a^2)=0$ as we did in the very first paragraph. And, similarly if $a=-b$ which is when $(a+b)^2=0$ we get just $a^2=0$. In either of those cases, we see that once again a and b would have to both be zero if $(a^2)+ab+(b^2)=0$.) And this second approach is more like what we would have to really do to prove Gelfand's assertion because it all boils down to the fact that $(a^2)+ab+(b^2)$ cannot be factored using real numbers (which is because it is always positive for nonzero a and/or b).

[Top](#) [Education Main Page](#)

Problem 125:

Recalling the infamous problem # 122(d), observe that $(a^3)+(b^3)+(b^3)-3abc = (a+b+c)*(some\ stuff)$. Then if $a+b+c=0$, $(a^3)+(b^3)+(b^3)-3abc = (a+b+c)*(some\ stuff) = 0*(some\ stuff) = 0$. Adding $3abc$ to both sides, we get $(a^3)+(b^3)+(b^3)=3abc$.

[Top](#) [Education Main Page](#)

Problem 126:

This problem really just amounts to a whole bunch of algebraic manipulations. Notice the $a+b+c$ appearing somewhere here, so we might be keeping a look out for the possibility of using some of our previous results. At any rate, I am just going to write down a series of equivalent expressions following each other one line after the next until I get something that looks good. So, it is implicit in what follows that if a given line is true, then the line that follows it must be true also:

$$\begin{aligned}
1/(a+b+c) &= (1/a) + (1/b) + (1/c) \\
1/(a+b+c) &= (bc/abc) + (ac/abc) + (ab/abc) \\
1/(a+b+c) &= (bc+ac+ab)/(abc) \\
abc/(a+b+c) &= (bc + ac + ab) \\
abc &= (bc + ac + ab)(a+b+c) \\
abc &= abc+(a^2)c+(a^2)b+(b^2)c+abc+a(b^2)+b(c^2)+a(c^2)+abc \\
0 &= abc+(b^2)c+a(c^2)+b(c^2)+(a^2)b+a(b^2)+(a^2)c+abc \\
0 &= [ab+(b^2)+ac+bc]c+a[ab+(b^2)+ac+bc] \\
0 &= (c+a)[(a+b)b+(a+b)c] \\
0 &= (c+a)(a+b)(b+c)
\end{aligned}$$

Some of that might have seemed like magic. Another approach might have been once we got to a bunch of one letter times another letter squared terms to just guess based on what we were trying to prove that it would all factor out to the product I have arrived at. (Then we would just try the product and see that it multiplies out to the mess we came upon in the middle.) At any rate, now that we have the last equation we can see that one of the terms must be zero so that $a=-b$, $b=-c$ or $c=-a$ which was what was to be proven.

[Top](#) [Education Main Page](#)

Problem 129:

Substituting a in for x , we see that in the first two terms we end up multiplying by $(a-a)=0$, so the only nonzero term is the third: $[(x-b)(x-c)]/[(a-b)(a-c)] = [(a-b)(a-c)]/[(a-b)(a-c)] = 1$ when $x=a$.

Substituting b in for x , we see that in the first and third terms we end up multiplying by $(b-b)=0$, so the only nonzero term is the second: $[(x-a)(x-c)]/[(b-a)(b-c)] = [(b-a)(b-c)]/[(b-a)(b-c)] = 1$ when $x=b$.

Substituting c in for x , we see that in the last two terms we end up multiplying by $(c-c)=0$, so the only nonzero term is the first: $[(x-a)(x-b)]/[(c-a)(c-b)] = [(c-a)(c-b)]/[(c-a)(c-b)] = 1$ when $x=c$.

[Top](#) [Education Main Page](#)

Problem 130:

If it takes a hours to fill one half a pool, then the pool is being filled at the rate of one half a pool per a hours. Similarly the other pipe fills the pool at the rate of one half a pool per b hours. When they are both turned on their rates simply add to each other so that they are together filling the pool at the rate of $(1/a)+(1/b)$ one half pool per hour, or $1/[(1/a)+(1/b)]$ hours to fill half a pool and twice as long to fill a whole pool:

$$2/[(1/a)+(1/b)]$$

$$= \frac{1}{\left\{\frac{1}{a} + \frac{1}{b}\right\}/2}$$

The very harmonic mean Gelfand spoke of!

[Top](#) [Education Main Page](#)

Problem 131:

The current helps the boat go down the river and slows the boat down every bit as much going back up the river. Now it is tempting to think that the current has a net effect of zero on the boat in a round trip, but we must remember that to travel the same distance at a slower rate takes longer. So, the current has a greater impact slowing the boat down than it does speeding it up. What is true though is that the boat would travel without a current at a rate equal to the average of the two velocities going with or against the current. So with the current the boat travels at a rate of $1/a$ trips per hour and at a rate of $1/b$ trips per hour without the current. Then, with no current the boat will travel at a rate of $\left\{\frac{1}{a} + \frac{1}{b}\right\}/2$ trips per hour. And so, it will take $1/\left\{\frac{1}{a} + \frac{1}{b}\right\}/2$ hours per trip without the current. My goodness, the harmonic mean again!

[Top](#) [Education Main Page](#)

Problem 132:

It is not entirely clear here what "half the trip" consists of. Gelfand, means half the *distance*. So, it takes $1/v(1)$ to travel the first half of the distance and $1/v(2)$ to travel the second half of the distance for a total time of $\left\{\frac{1}{v(1)} + \frac{1}{v(2)}\right\}$ to traverse the whole distance. Here I have assumed that the velocities are written in units of one half the distance of the trip per unit of time, so if we have gone two units of distance in the whole trip then the distance per unit time would be

$$\frac{2}{\left\{\frac{1}{v(1)} + \frac{1}{v(2)}\right\}} = \frac{1}{\left\{\frac{1}{v(1)} + \frac{1}{v(2)}\right\}/2}$$

You guessed it -- the Harmonic Mean.

[Top](#) [Education Main Page](#)

Problem 133:

(a)Based on the given information we have that $7 = x + (1/x)$, so that squaring both sides, we get

$$49 = \left[x + \frac{1}{x}\right]^2 = (x^2) + 2x\left(\frac{1}{x}\right) + \left(\frac{1}{x^2}\right)$$

$$= (x^2) + [1/(x^2)] + 2$$

Thus, subtracting 2 from both sides we see that $(x^2) + [1/(x^2)] = 49 - 2 = 47$.

(b) From the given information and part a, we get

$$\begin{aligned} 47 \cdot 7 &= \{(x^2) + [1/(x^2)]\} \cdot [x + (1/x)] \\ &= (x^3) + (1/x) + x + (1/x^3) \\ &= (x^3) + [1/(x^3)] + [x + (1/x)] \\ &= (x^3) + [1/(x^3)] + 7 \end{aligned}$$

Thus, subtracting 7 from both sides we see that $(x^3) + [1/(x^3)] = 47 \cdot 7 - 7 = 46 \cdot 7 = 322$.

[Top](#) [Education Main Page](#)

Problem 134:

The only way I know how to do this problem is using the Principle of Mathematical Induction. So, I am going to just give you an heuristic rationale for this assertion based on this principle (as opposed to the actual formal proof which I cannot seem to find a way to do with the tools we might have at this point). Consider the same calculation as in part (b) of the preceding problem only using n instead of 3. For $n > 2$, then, we have:

$$\begin{aligned} &([x^{(n-1)}] + [1/x^{(n-1)}]) \cdot [x + (1/x)] \\ &= (x^n) + [1/x^{(n-2)}] + [x^{(n-2)}] + [1/x^n] \end{aligned}$$

Subtracting $[1/x^{(n-2)}] + [x^{(n-2)}]$ from both sides yields: $(x^n) + [1/x^n] = ([x^{(n-1)}] + [1/x^{(n-1)}]) \cdot [x + (1/x)] - ([x^{(n-2)}] + [1/x^{(n-2)}])$. Or, in other words, we can always write the n^{th} expression in terms of the previous two expressions and the first expression. If I use the notation $a(n)$ to refer to the n^{th} expression, we have $a(n) = a(n-1) \cdot a(1) - a(n-2)$. And, based on part (a) of the previous problem $a(2) = a(1) - 1$. Here's where the heuristics come into play. Basically, if the first term is an integer, then the second term which is the difference of two integers is an integer. If all the terms up to the n^{th} term is an integer, then the $n+1$ term which is a product and a difference of integers must also be an integer. In a more formal setting, the base step is showing that the first two terms are integers and then using the strong principle of induction to show that if every case less than $k+1$ is an integer, then $k+1$ must be an integer. Then based on the (strong) principle of mathematical induction we could conclude that $a(n) = (x^n) + [1/x^n]$ is an integer for all n .

[Top](#) [Education Main Page](#)

Problem 135:

Using what he has already given us we have

$$\begin{aligned} & 1/(1+[(2x+1)/(3x+2)]) \\ &= \frac{(3x+2)}{(3x+2+2x+1)} \\ &= \frac{(3x+2)}{[(3+2)x+(2+1)]} \\ &= \frac{(3x+2)}{[5x+3]} \end{aligned}$$

Using what he has already given us we have

$$\begin{aligned} & 1/(1+[(3x+2)/(5x+3)]) \\ &= \frac{(5x+3)}{(5x+3+3x+2)} \\ &= \frac{(5x+3)}{[(5+3)x+(3+2)]} \\ &= \frac{(5x+3)}{[8x+5]} \end{aligned}$$

We just add the coefficients of the numerator to the denominator to get the denominator of the next term and use the denominator of the last term as the numerator of the next. The next in the series will be $(8x+5)/[(8+5)x+(5+3)] = (8x+5)/(13x+8)$. So, the sequence of coefficients goes 1, 1, 2, 3, 5, 8, 13, ... (add the last two coefficients to get the next one), and soem coefficeints lagg others, but they all follow this pattern.

[Top](#) [Education Main Page](#)

Problem 136:

The leading term will be $(2x)^5 = (2^5)(x^5) = 64x^5$ since the greatest power of x is obtained by multiplying the term with the greatest power of x inside the parentheses (i.e. $2x$) by itself 5 times.

[Top](#) [Education Main Page](#)

Problem 139:

The largest degree one can get is by taking the term with the largest degree from the expression being substituted and multiplying it by itself as many times as the largest degree of the original polynomial. So, the degree of the new polynomial in y after the substitution will be $10 \cdot 7 = 70$.

[Top](#) [Education Main Page](#)

Problem 143:

(a)

$$\begin{array}{r}
 x^2 + x + 1 \\
 \hline
 x - 1 \mid x^3 + 0x^2 + 0x - 1 \\
 -(x^3 - x^2) \\
 \hline
 x^2 + 0x \\
 -(x^2 - x) \\
 \hline
 x - 1 \\
 -(x - 1) \\
 \hline
 0
 \end{array}$$

(b)

$$\begin{array}{r}
 x^3 + x^2 + x + 1 \\
 \hline
 x - 1 \mid x^4 + 0x^3 + 0x^2 + 0x - 1 \\
 -(x^4 - x^3) \\
 \hline
 x^3 + 0x^2 \\
 -(x^3 - x^2) \\
 \hline
 x^2 + 0x \\
 -(x^2 - x) \\
 \hline
 x - 1 \\
 -(x - 1) \\
 \hline
 0
 \end{array}$$

(c)

$$\begin{array}{r}
 x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
 \hline
 x - 1 \mid x^{10} + 0x^9 + 0x^8 + 0x^7 + 0x^6 + 0x^5 + 0x^4 + 0x^3 + 0x^2 + 0x - 1 \\
 -(x^{10} - x^9) \\
 \hline
 x^9 + 0x^8 \\
 -(x^9 - x^8)
 \end{array}$$

$$\begin{array}{r}
 \text{-----} \\
 x^8 + 0x^7 \\
 -(x^8 - x^7) \\
 \text{-----} \\
 x^7 + 0x^6 \\
 -(x^7 - x^6) \\
 \text{-----} \\
 x^6 + 0x^5 \\
 -(x^6 - x^5) \\
 \text{-----} \\
 x^5 + 0x^4 \\
 -(x^5 - x^4) \\
 \text{-----} \\
 x^4 + 0x^3 \\
 -(x^4 - x^3) \\
 \text{-----} \\
 x^3 + 0x^2 \\
 -(x^3 - x^2) \\
 \text{-----} \\
 x^2 + 0x \\
 -(x^2 - x) \\
 \text{-----} \\
 x - 1 \\
 -(x - 1) \\
 \text{-----} \\
 0
 \end{array}$$

(d)

$$\begin{array}{r}
 x^2 - x + 1 \\
 \text{-----} \\
 x + 1 \mid x^3 + 0x^2 + 0x + 1 \\
 -(x^3 + x^2) \\
 \text{-----} \\
 -x^2 + 0x \\
 -(-x^2 - x) \\
 \text{-----} \\
 x + 1 \\
 -(x + 1)
 \end{array}$$

$$\begin{array}{r} \text{-----} \\ 0 \end{array}$$

(e)

$$\begin{array}{r} x^3 - x^2 + x - 1 \\ \text{-----} \\ x + 1 \mid x^4 + 0x^3 + 0x^2 + 0x + 1 \\ -(x^4 + x^3) \\ \text{-----} \\ -x^3 + 0x^2 \\ -(-x^3 - x^2) \\ \text{-----} \\ x^2 + 0x \\ -(x^2 + x) \\ \text{-----} \\ -x + 1 \\ -(-x - 1) \\ \text{-----} \\ 2 \end{array}$$

[Top](#) [Education Main Page](#)

Problem 145:

(a) $x^n - 1$ is always divisible by $x - 1$ because 1 is always a root -- just plug it in: $1^n - 1 = 1 - 1 = 0$.

(b) When is -1 a root of $x^n + 1$? When $(-1)^n + 1 = 0$ or when $(-1)^n = -1$ which is true if and only if n is odd.

[Top](#) [Education Main Page](#)

Problem 146:

(a) Try plugging in 1 for x to see if it is a root. You get $(1)^4 + 5(1) - 6 = 1 + 5 - 6 = 0$. Then $x - 1$ is a factor so that we can divide it into $x^4 + 5x - 6$:

$$\begin{array}{r} x^3 + x^2 + x + 6 \\ \text{-----} \\ x - 1 \mid x^4 + 0x^3 + 0x^2 + 5x - 6 \end{array}$$

$$-(x^4 - x^3)$$

$$x^3 + 0x^2$$

$$-(x^3 - x^2)$$

$$x^2 + 5x$$

$$-(x^2 - x)$$

$$6x - 6$$

$$-(6x - 6)$$

$$0$$

Thus, $x^4 + 5x - 6 = (x-1)(x^3 + x^2 + x + 6)$

(b) Because nothing is subtracted, you can tell any root will have to be some negative number (to get the sum to add to zero when you plug it in for x). Try plugging in -1 for x to see if it is a root. You get $(-1)^4 + 3(-1)^2 + 5(-1) + 1 = 1 + 3 - 5 + 1 = 0$. Then, $x - (-1) = x + 1$ is a factor so that we can divide it into $x^4 + 3x^2 + 5x + 1$:

$$x^3 - x^2 + 4x + 1$$

$$x + 1 \mid x^4 + 0x^3 + 3x^2 + 5x + 1$$

$$-(x^4 + x^3)$$

$$-x^3 + 3x^2$$

$$-(-x^3 - x^2)$$

$$4x^2 + 5x$$

$$-(4x^2 + 4x)$$

$$x + 1$$

$$-(x + 1)$$

$$0$$

Thus, $x^4 + 3x^2 + 5x + 1 = (x+1)(x^3 - x^2 + 4x + 1)$

(c) Casual inspection tells us that 1 is not a root, so try -1 again. You get $(-1)^3 - 3(-1) - 2 = -1 + 3 - 2 = 0$. Then $x + 1$ is a factor so that we can divide it into $x^3 - 3x - 2$:

$$x^2 - x - 2$$

$$\begin{array}{r}
 \text{-----} \\
 x + 1 \mid x^3 + 0x^2 - 3x - 2 \\
 -(x^3 + x^2) \\
 \text{-----} \\
 -x^2 - 3x \\
 -(-x^2 - x) \\
 \text{-----} \\
 2x - 2 \\
 -(-2x - 2) \\
 \text{-----} \\
 0
 \end{array}$$

Thus, $x^3-3x-2 = (x+1)(x^2-x-2)$

[Top](#) [Education Main Page](#)

Problem 150:

We were told how to do this in Gelfand's solution to problem 149. And, we have seen on several occasions now that $x-1$ always divides $(x^n)-1$ so that 1 is always a root of $(x^n)-1$. So, all that remains to be seen is when -1 is a root of $(x^n)-1$, or in other words, when $[(-1)^n]-1 = 0$ or $(-1)^n = 1$ which happens whenever n is even (and is not true otherwise). So, $(x^n)-1$ is divisible by $(x^2)-1$ whenever n is even, and otherwise it is not divisible by $(x^2)-1$.

[Top](#) [Education Main Page](#)

Problem 151:

He introduces the notation $P(x)$ in the next section. I will just use it here for consistency and convenience. Plug these values in for x and you will get nothing but the remainder left over:

$$\begin{array}{rcl}
 P(1) & = & [(1^2)-1](\text{quotient})+[a*(1)+b] \\
 P(1) & = & 0*(\text{quotient})+(a+b) \\
 P(1) & = & a+b
 \end{array}$$

Similarly,

$$\begin{array}{rcl}
 P(-1) & = & [((-1)^2)-1](\text{quotient})+[a*(-1)+b] \\
 P(-1) & = & 0*(\text{quotient})+(-a+b) \\
 P(-1) & = & b-a
 \end{array}$$

So, if you know what $P(1)$ and $P(-1)$ is, you can figure out what a and b are which determines what the remainder is. If you solve for these equations, you will get

$$\begin{aligned} a &= [P(1)-P(-1)]/2 \\ b &= [P(1)+P(-1)]/2 \end{aligned}$$

[Top](#) [Education Main Page](#)

Problem 152:

We are given that

$$P = [(x^2)-1](\text{quotient})+(5x-7).$$

Then, we can factor

$$(x^2)-1 = (x-1)(x+1),$$

so that

$$P = (x-1)(x+1)(\text{quotient})+(5x-7).$$

So, if we divide by $x-1$, we get

$$\begin{aligned} P/(x-1) &= [(x-1)(x+1)(\text{quotient})+(5x-7)]/(x-1) \\ &= (x-1)(x+1)(\text{quotient})/(x-1) + (5x-7)/(x-1) \\ &= (x+1)(\text{quotient}) + (5x-7)/(x-1). \end{aligned}$$

Thus, the remainder when dividing P by $x-1$ will be the same as the remainder when dividing $5x-7$ by $x-1$ which is true essentially because $x-1$ divides $(x^2)-1$. So, the remainder is $-7-(-5) = -2$.

[Top](#) [Education Main Page](#)

Problem 153:

I will use a , b , and c to denote the roots rather than using the subscripts as Gelfand does. Thus, $P(a) = P(b) = P(c) = (c^3)+(c^2)-10c+1 = 0$. Also, the key issue in this problem is that the polynomial must be a *polynomial* and it must have *integer* coefficients.

(a) Try to undo (in advance) in the polynomial whatever it is that Gelfand does to the roots. So, if he has added one to each root, then subtract one from x in P so that when you substitute any one of the new roots into the new polynomial, it will just simplify to P with the old root substituted. So, in this case try

the polynomial $Q(x) = P(x-1) = [(x-1)^3] + [(x-1)^2] - 10(x-1) + 1$. Then, $Q(a+1) = [(a+1-1)^3] + [(a+1-1)^2] - 10(a+1-1) + 1 = (a^3) + (a^2) - 10a + 1 = P(a) = 0$. And the same calculation will occur for $Q(b+1)$ and $Q(c+1)$ (so that $a+1$, $b+1$ and $c+1$ are all roots of Q). Finally if we multiply $Q(x)$ out we get

$$\begin{aligned} Q(x) &= [(x-1)^3] + [(x-1)^2] - 10(x-1) + 1 \\ &= [(x^3) - 3(x^2) + 3x - 3] + [(x^2) - 2x + 1] - (10x - 10) + 1 \\ &= (x^3) - 3(x^2) + 3x - 3 + (x^2) - 2x + 1 - 10x + 10 + 1 \\ &= (x^3) + [-3(x^2) + (x^2)] + (-2x - 10x + 3x) + (10 + 1 + 1 - 3) \\ &= (x^3) - 2(x^2) + 9x + 9 \end{aligned}$$

which has integer coefficients. The reason this all worked out was because simply adding or subtracting integers from x (or even multiplying x by an integer) in a polynomial with integer coefficients results in another polynomial with integer coefficients. (This is so because if we take some integers and add them, subtract them or multiply them together, we always end up with another integer.)

(b) We cannot just willy nilly undo in advance the multiplication of 2 because we will have to divide by 2 and end up with a polynomial, but probably not one with *integer* coefficients. We can solve this problem by multiplying the whole polynomial by a sufficiently large power of 2 to cancel any 2's we might end up dividing by. Also, such a multiplication will not affect the roots. Therefore, since the largest power of $1/2$ will be 3, try multiplying by 2^3 to get $Q(x) = (2^3)P(x/2) = (2^3)*\{[(x/2)^3] + [(x/2)^2] - 10(x/2) + 1\}$. Of course, now when we plug in $2a$, for instance, we will get $Q(2a) = (2^3)*\{[(2a/2)^3] + [(2a/2)^2] - 10(2a/2) + 1\} = (2^3)*\{[(a)^3] + [(a)^2] - 10(a) + 1\} = (2^3)*P(a) = 0$. And the same sort of result will occur for $Q(2b)$ and $Q(2c)$. Finally, multiplying $Q(x)$ out we get

$$\begin{aligned} Q(x) &= (2^3)*\{[(x/2)^3] + [(x/2)^2] - 10(x/2) + 1\} \\ &= (2^3)*[(x^3)/(2^3)] + (2^3)*[(x^2)/(2^2)] - (2^3)*10(x/2) + (2^3)*1 \\ &= [(2^3)/(2^3)]*(x^3) + [(2^3)/(2^2)]*(x^2) - [(2^3)/2]*10x + (2^3) \\ &= 1*(x^3) + 2*(x^2) - (2^2)*10x + (2^3) \\ &= (x^3) + 2(x^2) - 40x + 8 \end{aligned}$$

which has integer coefficients.

(c) This time we are going to do something similar by undoing in advance what has been done to the root which we can do by substituting $1/x$ for x in our polynomial. However, if we do that we end up with an expression that has negative powers of x . To undo that outcome without eliminating the roots, multiply the whole expression by the x raised to largest power of $1/x$. Therefore, try $Q(x) = (x^3)P(1/x) = (x^3)*\{[(1/x)^3] + [(1/x)^2] - 10(1/x) + 1\}$. Once again observe that $Q(1/a) = [(1/a)^3]*P(1/(1/a)) = [(1/a)^3]*P(a) = 0$. So, $1/a$ is a root, and similarly, $1/b$ and $1/c$ are roots as well. Now, is $Q(x)$ even a polynomial? Let's see:

$$\begin{aligned} Q(x) &= (x^3)*\{[(1/x)^3] + [(1/x)^2] - 10(1/x) + 1\} \\ &= (x^3)*[1/(x^3)] + (x^3)*[1/(x^2)] - (x^3)*10(1/x) + (x^3)*1 \\ &= [(x^3)/(x^3)] + [(x^3)/(x^2)] - 10*(x^3)/x + (x^3) \\ &= 1 + x - 10(x^2) + (x^3) \end{aligned}$$

which is, indeed, a polynomial with integer coefficients.

[Top](#) [Education Main Page](#)

Problem 154:

Observe that the quadratic can be factored: $(x^2)-3x+2 = (x-1)(x-2)$. So, 1 and 2 are roots of the quadratic and since it divides the cubic, 1 and 2 must also be roots of the cubic. That is, $0 = (1^3)+a(1^2)+1+b = 2+a+b$ or $a+b = -2$. And, $0 = (2^3)+a(2^2)+2+b = 10+4a+b$ or $4a+b = -10$. Then, $a = -2-b$ from the first equation and substituting that into the second equation we get $-10 = 4(-2-b)+b = -8-4b+b = -8-3b$ or $3b = 2$ or $b = 2/3$. Now, back to the first equation, $a = -2-b = -2-(2/3) = -8/3$. (He doesn't ever really teach us how to solve a system of two equations in two unknowns or even a linear equation at this point in the book, but it seems he thinks we can just be clever enough to do it. Inventing the method of substitution on one's own may not be terribly far out of the realm of possibilities. If it is, then you'll have to just teach that real quick from some other supplemental text or just skip the problems that seem to rely on such a thing and possibly come back to them later.)

[Top](#) [Education Main Page](#)

Problem 155:

$$P(0) = (0^3)-2 = -2$$

$$P(1) = (1^3)-2 = -1$$

$$P(2) = (2^3)-2 = 6$$

$$P(3) = (3^3)-2 = 25$$

$$P(4) = (4^3)-2 = 62$$

$$P(5) = (5^3)-2 = 123$$

$$P(6) = (6^3)-2 = 214$$

[Top](#) [Education Main Page](#)

Problem 156:

In order to solve this problem we will need to evaluate the polynomial in a general way at four successive values:

$$P(x) = (x^2)-x-4$$

$$P(x+1) = [(x+1)^2]-(x+1)-4 = [(x^2)+2x+1]-x-1-4 = (x^2)+x-4$$

$$P(x+2) = [(x+2)^2] - (x+2) - 4 = [(x^2) + 4x + 4] - x - 2 - 4 = (x^2) + 3x - 2$$

$$P(x+3) = [(x+3)^2] - (x+3) - 4 = [(x^2) + 6x + 9] - x - 3 - 4 = (x^2) + 5x + 2$$

Thus,

$$P(x+1) - P(x) = [(x^2) + x - 4] - [(x^2) - x - 4] = 2x$$

and

$$P(x+3) - P(x+2) = [(x^2) + 5x + 2] - [(x^2) + 3x - 2] = 2x + 2.$$

Then, the second difference is

$$[P(x+3) - P(x+2)] - [P(x+1) - P(x)] = [2x + 2] - [2x] = 2.$$

And so, regardless of what values we plug in for x , the second differences starting at that value will always equal 2.

[Top](#) [Education Main Page](#)

Problem 157:

Do this the same way we just did the last one, only now we need some more general coefficients (a , b , and c , for instance):

$$P(x) = a(x^2) + bx + c$$

$$P(x+1) = a[(x+1)^2] + b(x+1) + c = a[(x^2) + 2x + 1] + bx + b + c = a(x^2) + (2a+b)x + (a+b+c)$$

$$P(x+2) = a[(x+2)^2] + b(x+2) + c = a[(x^2) + 4x + 4] + bx + 2b + c = a(x^2) + (4a+b)x + (4a+2b+c)$$

$$P(x+3) = a[(x+3)^2] + b(x+3) + c = a[(x^2) + 6x + 9] + bx + 3b + c = a(x^2) + (6a+b)x + (9a+3b+c)$$

Thus,

$$\begin{aligned} P(x+1) - P(x) &= [a(x^2) + (2a+b)x + (a+b+c)] - [a(x^2) + bx + c] \\ &= 2ax + a + b \end{aligned}$$

and

$$\begin{aligned} P(x+3) - P(x+2) &= [a(x^2) + (6a+b)x + (9a+3b+c)] - [a(x^2) + (4a+b)x + (4a+2b+c)] \\ &= 2ax + 5a + b. \end{aligned}$$

Then, the second difference is

$$[P(x+3)-P(x+2)] - [P(x+1)-P(x)] = [2ax+5a+b] - [2ax+a+b] = 4a.$$

And so, regardless of what values we plug in for x , the second differences starting at that value will always equal $4a$.

[Top](#) [Education Main Page](#)

Problem 158:

I'm just going to go ahead and do it all the way out just like I did the quadratic (this time using a , b , c and d):

$$P(x) = a(x^3) + b(x^2) + cx + d$$

$$\begin{aligned} P(x+1) &= a[(x+1)^3] + b[(x+1)^2] + c(x+1) + d \\ &= a[(x^3) + 3(x^2) + 3x + 1] + b[(x^2) + 2x + 1] + c(x+1) + d \\ &= a(x^3) + (3a+b)(x^2) + (3a+2b+c)x + (a+b+c+d) \end{aligned}$$

$$\begin{aligned} P(x+2) &= a[(x+2)^3] + b[(x+2)^2] + c(x+2) + d \\ &= a[(x^3) + 6(x^2) + 12x + 8] + b[(x^2) + 4x + 4] + c(x+2) + d \\ &= a(x^3) + (6a+b)(x^2) + (12a+4b+c)x + (8a+4b+2c+d) \end{aligned}$$

$$\begin{aligned} P(x+3) &= a[(x+3)^3] + b[(x+3)^2] + c(x+3) + d \\ &= a[(x^3) + 9(x^2) + 27x + 27] + b[(x^2) + 6x + 9] + c(x+3) + d \\ &= a(x^3) + (9a+b)(x^2) + (27a+6b+c)x + (27a+9b+3c+d) \end{aligned}$$

$$\begin{aligned} P(x+4) &= a[(x+4)^3] + b[(x+4)^2] + c(x+4) + d \\ &= a[(x^3) + 12(x^2) + 48x + 64] + b[(x^2) + 8x + 16] + c(x+4) + d \\ &= a(x^3) + (12a+b)(x^2) + (48a+8b+c)x + (64a+16b+4c+d) \end{aligned}$$

$$\begin{aligned}
 P(x+5) &= a[(x+5)^3] + b[(x+5)^2] + c(x+5) + d \\
 &= a[(x^3) + 15(x^2) + 75x + 125] + b[(x^2) + 10x + 25] + c(x+5) + d \\
 &= a(x^3) + (15a+b)(x^2) + (75a+10b+c)x + (125a+25b+5c+d)
 \end{aligned}$$

$$\begin{aligned}
 P(x+6) &= a[(x+6)^3] + b[(x+6)^2] + c(x+6) + d \\
 &= a[(x^3) + 18(x^2) + 108x + 216] + b[(x^2) + 12x + 36] + c(x+6) + d \\
 &= a(x^3) + (18a+b)(x^2) + (108a+12b+c)x + (216a+36b+6c+d)
 \end{aligned}$$

$$\begin{aligned}
 P(x+7) &= a[(x+7)^3] + b[(x+7)^2] + c(x+7) + d \\
 &= a[(x^3) + 21(x^2) + 147x + 343] + b[(x^2) + 14x + 49] + c(x+7) + d \\
 &= a(x^3) + (21a+b)(x^2) + (147a+14b+c)x + (343a+49b+7c+d)
 \end{aligned}$$

Then the first differences are:

$$\begin{aligned}
 P(x+1) - P(x) &= [a(x^3) + (3a+b)(x^2) + (3a+2b+c)x + (a+b+c+d)] \\
 &\quad - [a(x^3) + b(x^2) + cx + d]
 \end{aligned}$$

$$= 3a(x^2) + (3a+2b)x + (a+b+c)$$

$$\begin{aligned}
 P(x+3) - P(x+2) &= [a(x^3) + (9a+b)(x^2) + (27a+6b+c)x + (27a+9b+3c+d)] \\
 &\quad - [a(x^3) + (6a+b)(x^2) + (12a+4b+c)x + (8a+4b+2c+d)]
 \end{aligned}$$

$$= 3a(x^2) + (15a+2b)x + (19a+5b+c)$$

$$\begin{aligned}
 P(x+5) - P(x+4) &= [a(x^3) + (15a+b)(x^2) + (75a+10b+c)x + (125a+25b+5c+d)] \\
 &\quad - [a(x^3) + (12a+b)(x^2) + (48a+8b+c)x + (64a+16b+4c+d)]
 \end{aligned}$$

$$= 3a(x^2) + (27a+2b)x + (61a+9b+c)$$

$$P(x+7) - P(x+6) = [a(x^3) + (21a+b)(x^2) + (147a+14b+c)x + (343a+49b+7c+d)] \\ - [a(x^3) + (18a+b)(x^2) + (108a+12b+c)x + (216a+36b+6c+d)]$$

$$= 3a(x^2) + (39a+2b)x + (127a+13b+c)$$

The second differences are:

$$\begin{aligned} & [P(x+3)-P(x+2)] - [P(x+1)-P(x)] \\ = & [3a(x^2)+(15a+2b)x+(19a+5b+c)] - [3a(x^2)+(3a+2b)x+(a+b+c)] \\ = & 12ax+18a+4b \end{aligned}$$

and

$$\begin{aligned} & [P(x+7)-P(x+6)] - [P(x+5)-P(x+4)] \\ = & [3a(x^2)+(39a+2b)x+(127a+13b+c)] - [3a(x^2)+(27a+2b)x+(61a+9b+c)] \\ = & 12ax+66a+4b \end{aligned}$$

So finally the third difference is

$$\begin{aligned} & \{[P(x+7)-P(x+6)]-[P(x+5)-P(x+4)]\} - \{[P(x+3)-P(x+2)]-[P(x+1)-P(x)]\} \\ = & (12ax+66a+4b) - (12ax+18a+4b) \\ = & 48a \end{aligned}$$

Or in other words, for third degree polynomials, it is their *third* differences that are all the same.

[Top](#) [Education Main Page](#)

Problem 159:

x	P(x)
1	(1 ²)+1+41 = 43
2	(2 ²)+2+41 = 47
3	(3 ²)+3+41 = 53

4	$(4^2)+4+41 = 61$
5	$(5^2)+5+41 = 71$
6	$(6^2)+6+41 = 83$
7	$(7^2)+7+41 = 97$
8	$(8^2)+8+41 = 113$
9	$(9^2)+9+41 = 131$
10	$(10^2)+10+41 = 151$

To check one can simply try to divide each number by each prime less than $1/2$ of it. I will do this for 43...

p	43/p
2	$43/2 = 21.5$
3	$43/3 = 14.3...$
5	$43/5 = 8.6$
7	$43/7 = 6.1...$
11	$43/11 = 3.9...$
13	$43/13 = 3.3...$
17	$43/17 = 2.5...$
19	$43/19 = 2.2...$
23	$43/23 = 1.9...$

So, 43 has to be prime because any other number that might divide it is either a product of the p's in the above table or is so large that it would divide into 43 less than twice. One can check that the other numbers are prime, in the same fashion. However, if nothing else, once one gets up to $x=40$, the numbers will no longer be prime because

$$\begin{aligned}
 &P(40) \\
 &= (40^2)+40+41 \\
 &= 40*(40+1)+41 \\
 &= 40*41+41 \\
 &= (40+1)*41 \\
 &= 41^2
 \end{aligned}$$

which most definitely is not prime.

[Top](#) [Education Main Page](#)

Problem 161:

If $P(x)$ is a polynomial of degree at most one, then we can express it $P(x) = ax+b$. Thus, $0 = P(1) = a*(1)+b$ or $a = -b$. Also, $0 = P(2) = a*(2)+b$ or $2a = -b$. Then, $2a = -b = a$ or, subtracting a from both sides we get that $a = 0$ which in turn implies that $-b = a = 0$ or $b = 0$. Therefore, $P(x) = ax+b = 0x+0 = 0$.

[Top](#) [Education Main Page](#)

Problem 165:

What Gelfand wants you to recognize here is that we are dealing with three points of the polynomial $P(x) = a(x^2)+bx+c$. The equations are just $P(4)=0$, $P(7)=0$, and $P(10)=0$, respectively. That is $P(x)$ has three roots which is impossible unless its degree is 3 or more or $P(x)=0$ (following the same argument as in Problem 164). Since the degree of $P(x)$ is less than three, $P(x)=0$.

Another way to do this is to just solve this system of equations. Since they are all equal to zero, they are equal to each other: $16a+4b+c = 49a+7b+c = 100a+10b+c$. Subtracting c , we get that $16a+4b = 49a+7b = 100a+10b$. Looking at the first equation and collecting like terms we have that $33a+3b = 0$ or $3b = -33a$ or $b = -11a$. Now substitute $-11a$ in for b in the second equation to get $49a+7(-11a) = 100a+10(-11a)$ or $(49-77)a = (100-110)a$ or, collecting like terms $0 = (77-49+100-110)a = 11a$ which implies that a must equal zero. Then, $b = -11a = -11*0 = 0$. If $a = b = 0$, then $0 = 16a+4b+c = 16*0+4*0+c = c$, and so $a=b=c=0$.

[Top](#) [Education Main Page](#)

Problem 166:

GOD! This is a great problem! I can really see a school aged child doing it in this context, on the one hand, and on the other this is a real proof. Though this is not necessarily a hard problem by graduate student standards (depending on the context you see it in), it is very similar to the sort of thing one might see and the sort of technique one might use in graduate level mathematics. (For instance, it appears in Chapter 9 of this [book](#).) AND, as it appears it is a generalization of a more specific result. AND, the student is learning it the way they would in graduate school -- by mimicking other proofs (specifically the specific result it is a generalization of), so let's try to mimic Gelfand's approach for problem 164. Let $P(x)$ and $Q(x)$ be two polynomials of degree n and let $R(x) = P(x) - Q(x)$. Let $x(1), x(2), \dots, x(n), x(n+1)$ be some numbers so that $P(x(1))=Q(x(1))$, $P(x(2))=Q(x(2))$, ..., $P(x(n))=Q(x(n))$, $P(x(n+1))=Q(x(n+1))$. Then, $R(x(1))=R(x(2))=\dots=R(x(n))=R(x(n+1))=0$, so that $x(1), x(2), \dots, x(n), x(n+1)$ are roots of R . But, R can have at most n roots or it is identically equal to zero for all x . Thus, $0 = R(x) = P(x) - Q(x)$ or $P(x) = Q(x)$ for all x . Or in other words, whenever two n degree polynomials are equal on $n+1$ or more points, they are the same polynomial.

[Top](#) [Education Main Page](#)

Problem 168:

Following the same approach as in the preceding problem, construct several polynomials that have roots at all the values but one where it actually attains the desired value. Then, the polynomial that attains all the values at the given points is simply the sum of these other polynomials.

So, first we construct a polynomial with roots of 0, 1, and 2 but that equals 2 and $x = -1$. Thus we write $Q(x) = a(x-0)(x-1)(x-2)$ and solve for a in $2 = Q(-1) = a(-1)(-2)(-3) = -6a$. Thus, $a = -2/6 = -1/3$, so that $Q(x) = -x(x-1)(x-2)/3$

Next, we construct a polynomial with roots of -1, 1, and 2 but that equals 1 and $x = 0$. Thus we write $R(x) = a[x-(-1)](x-1)(x-2)$ and solve for a in $1 = R(0) = a(1)(-1)(-2) = 2a$. Thus, $a = 1/2$, so that $R(x) = (x+1)(x-1)(x-2)/2$

Now, construct a polynomial with roots of -1, 0, and 2 but that equals 2 and $x = 1$. Thus we write $S(x) = a[x-(-1)](x-0)(x-2)$ and solve for a in $2 = S(1) = a(2)(1)(-1) = -2a$. Thus, $a = -2/2 = -1$, so that $S(x) = -x(x+1)(x-2)$

Finally, construct a polynomial with roots of -1, 0, and 1 but that equals 7 and $x = 2$. Thus we write $T(x) = a[x-(-1)](x-0)(x-1)$ and solve for a in $7 = T(2) = a(3)(2)(1) = 6a$. Thus, $a = 7/6$, so that $T(x) = 7x(x+1)(x-1)/6$

And so our polynomial $P(x) = Q(x) + R(x) + S(x) + T(x) = -x(x-1)(x-2)/3 + (x+1)(x-1)(x-2)/2 - x(x+1)(x-2) + 7x(x+1)(x-1)/6$ which is a polynomial of degree three. Just to check our answer:

$P(-1)$	=	$(-2)(-3)/3$	+	0	-	0	+	0	=	2
$P(0)$	=	0	+	$(1)(-1)(-2)/2$	-	0	+	0	=	1
$P(1)$	=	0	+	0	-	$(1)(2)(-1)$	+	0	=	2
$P(2)$	=	0 + 0 - 0 + 7(2)(3)(1)/6	= 7							

[Top](#) [Education Main Page](#)

Problem 169:

This follows immediately from problem 166 with $n = 9$ (and just sayign that would be sufficient for this problem). But, just to relive the glory of a real proof, let us do it again with $n = 9$. Suppose $Q(x)$ is a polynomial of degree no greater than 9 such that $Q(x(1)) = y(1)$, $Q(x(2)) = y(2)$, ..., $Q(x(10)) = y(10)$, and let $R(x) = P(x) - Q(x)$. $R(x)$ is a polynomial of degree no greater than 9 with $R(x(1)) = P(x(1)) - Q(x(1)) = y(1) - y(1) = 0$, ..., $R(x(10)) = P(x(10)) - Q(x(10)) = y(10) - y(10) = 0$. That is, $y(1)$, ..., $y(10)$ are all roots of $R(x)$, but that is impossible since $R(x)$ is of degree less than 10 unless $R(x) = 0$ for all x . Then, $P(x) - Q(x) = 0$ or $P(x) = Q(x)$ for all x . Or in other words, there is only one polynomial of degree no greater than 9 that attains such values.

[Top](#) [Education Main Page](#)

Problem 170:

Similar to problem 165, Gelfand wants us to observe that these equations amount to $P(10)=18.37$, $P(6)=0.05$, and $P(2)=-3$, respectively, where $P(x) = a(x^2)+bx+c$. We know thus far that 3 values for a polynomial of degree 2 is sufficient to *uniquely* determine it, but we don't actually know that three values cannot *over* determine it. It may be that for some sets of values there are *no* polynomials that attain them. So, probably the best way to do this is to construct a polynomial of degree 2 that meets these conditions without actually solving for everything. So we can just observe that there exists polynomials $R(x)$, $S(x)$, and $Q(x)$ such that:

$$\begin{array}{llllllll} R(10) & = & 18.37, & R(6) & = & 0, & \text{and} & R(2) & = & 0 \\ S(10) & = & 0, & S(6) & = & 0.05, & \text{and} & S(2) & = & 0 \\ Q(10) & = & 0, & Q(6) & = & 0, & \text{and} & Q(2) & = & -3 \end{array}$$

Moreover, these polynomials require only two roots so need not have degree greater than 2. Supposing that they do not, then $P(x) = Q(x) + R(x) + S(x)$ is a polynomial of degree no greater than 2 such that $P(10)=18.37$, $P(6)=0.05$, and $P(2)=-3$. Thus, a , b , and c do indeed exist so that the equations shown hold - they are the coefficients of $P(x)$.

[Top](#) [Education Main Page](#)

Problem 175:

Similar to the preceding problem the k^{th} term is $2k$ (the 1st is $2=2*1$, the 2nd is $4=2*2$, the 3rd is $6=2*3$, the 4th is $8=2*4$, etc). Then the 1000th term will be $2*1000=2000$.

[Top](#) [Education Main Page](#)

Problem 176:

Each term is just one less than the corresponding term in the progression of the previous problem. So if the 100th term in the previous problem is 2000, then the 100th term in this one will be $2000-1=1999$.

[Top](#) [Education Main Page](#)

Problem 178:

Whereas we had to add d to go from one term to the next, in reverse order we must go from what was originally the next term to the one that originally preceded it. Thus we must subtract d in every case to go from one term to the next in the progression written in reverse order. So, yes, such a progression is still an arithmetic progression with difference $-d$.

Problem 179:

The new progression is every other term of the original progression. To get from one term to the next we must first add d to get the term we are skipping and then add d again to get to the next term. That is, for any given term in the progression we must add $2d$ to get to the next term, and so such a progression is still an arithmetic progression with difference $2d$.

Problem 180:

In the new sequence, to get from the first to the second term, we must add d , but to get from the second to the third term, we must add $2d$. So, there is not a *fixed* difference, and so this new progression is not an *arithmetic* progression.

Problem 183:

The terms are a , $a+d$, $(a+d)+d$, b and so $b = [(a+d)+d]+d = a + 3d$ where d is the fixed difference. Solving for d , we get that $d = (b-a)/3$, and so the terms in our progression are a , $a+(b-a)/3$, $a+2(b-a)/3$, b which can be simplified/rearranged into a , $(2a+b)/3$, $(a+2b)/3$, b .

Problem 184:

Using the first hint, we solve for n where $2n-1=999$. Thus, $2n=1000$ or $n=500$, so that there are $n=500$ terms in a progression going from the 1st to the n^{th} term. The second hint amounts to solving for $2n=1000$ (the same thing) by observing that every term in our original progression is one less than the terms in this other progression in which the n^{th} term is equal to $2n$. Since both progressions have the same number of terms, we can just find how many terms this other progression has which amounts to solving for n in $2n=1000$.

Problem 188:

We have seen in previous problems that the n^{th} term of the first n odd numbers is equal to $2n-1$. The first term is simply 1. Thus using 186 with $a=1$ and $b=2n-1$ we get that the sum of the first n odd numbers is $n(a+b)/2 = n[1+(2n-1)]/2 = n(2n)/2 = n^2$, a perfect square.

[Top](#) [Education Main Page](#)

Problem 194:

By starting with two instead of one, you've just spared yourself a minute since one minute after starting with one, you will be in exactly the same situation as starting with two, namely, with a vessel containing two bacteria. So, it takes $30-1=29$ minutes if you start with two.

[Top](#) [Education Main Page](#)

Problem 196:

The new progression is every other term of the original progression. To get from one term to the next we must first multiply by q to get the term we are skipping and then multiply by q again to get to the next term. That is, for any given term in the progression we must multiply by q^2 to get to the next term, and so such a progression is still a geometric progression with common ratio q^2 .

[Top](#) [Education Main Page](#)

Problem 197:

In the new sequence, to get from the first to the second term, we must multiply by q , but to get from the second to the third term, we must multiply by q^2 . So, there is not a *fixed* ratio, and so this new progression is not a *geometric* progression.

[Top](#) [Education Main Page](#)

Problem 201:

$1+(1/2)$		$=$		$3/2$
$1+(1/2)+(1/4)$	$=$	$(3/2)+(1/4)$	$=$	$7/4$
$1+(1/2)+(1/4)+(1/8)$	$=$	$(7/4)+(1/8)$	$=$	$15/8$
$1+\dots+(1/8)+(1/16)$	$=$	$(15/8)+(1/16)$	$=$	$31/16$

$1+\dots+(1/16)+(1/32)$	$=$	$(31/16)+(1/32)$	$=$	$63/32$
$1+\dots+(1/32)+(1/64)$	$=$	$(63/32)+(1/64)$	$=$	$127/64$
$1+\dots+(1/64)+(1/128)$	$=$	$(127/64)+(1/128)$	$=$	$255/128$
$1+\dots+(1/128)+(1/256)$	$=$	$(255/128)+(1/256)$	$=$	$511/256$
$1+\dots+(1/256)+(1/512)$	$=$	$(511/256)+(1/512)$	$=$	$1023/512$
$1+\dots+(1/512)+(1/1024)$	$=$	$(1023/512)+(1/1024)$	$=$	$2027/1024$
...				
$1+\dots+[1/(2^n)] = \{[2^{(n+1)}]-1\}/(2^n)$				

[Top](#) [Education Main Page](#)

Problem 204:

Gelfand really gives you the answer to this problem. Trying $d = 1/(2*3*5) = 1/30$ and starting at the smallest one, we get:

$$\begin{aligned}
 5/30 + 1/30 &= 6/30 = 1/5 \\
 6/30 + 1/30 &= 7/30 \\
 7/30 + 1/30 &= 8/30 \\
 8/30 + 1/30 &= 9/30 \\
 9/30 + 1/30 &= 10/30 = 1/3 \\
 10/30 + 1/30 &= 11/30 \\
 11/30 + 1/30 &= 12/30 \\
 12/30 + 1/30 &= 13/30 \\
 13/30 + 1/30 &= 14/30 \\
 14/30 + 1/30 &= 15/30 = 1/2
 \end{aligned}$$

[Top](#) [Education Main Page](#)

Problem 206:

We assume that n or m is *non-negative* (actually -- not *positive*) because our argument is based on the divisibility of integers by 2, 3, or 5. So, if m were negative for instance, then $(2^m)*(5^n)$ would not be "even" since it wouldn't even be an integer. I suppose we could handle it in 4 cases, then: n and m are nonnegative (as was done), n is nonnegative and m is nonpositive, n is nonpositive and m is nonnegative, and n and m are both nonpositive. In the second case, we rewrite $3^{(m+n)} = (2^m)*(5^n)$ as $(2^{[-m]})*(3^n) = (3^{[-m]})*(5^n)$ and observe that if m is not zero then the expression on the left hand side is divisible by 2 while the expression on the right hand side clearly is not. So, $m=0$, and the argument follows. We rewrite the equation to get $(5^{[-n]})*(3^m) = (2^m)*(3^{[-n]})$ where n is nonpositive and m is nonnegative, proceeding in a similar manner from here as in the previous case.

Finally in the case where both n and m are nonpositive we simply invert everything to get $3^{(-m-n)} = (2^{[-m]}) \cdot (5^{[-n]})$, and the same arguments apply here as in the first case.

[Top](#) [Education Main Page](#)

Problem 211:

Using the sequence suggested in the hint, the n^{th} term is $a+(n-1)\sqrt{2}$ where a is the initial term, zero. Thus, the first term is an integer (zero) and any term other than the first has the form $k\sqrt{2}$ for some integer $k > 0$. If $k\sqrt{2}$ where an integer, m say, then $\sqrt{2} = m/k$ would be rational which, as we shall soon see, it is not. Thus, $k\sqrt{2}$ can never be equal to an integer for positive (indeed, nonzero) k .

[Top](#) [Education Main Page](#)

Problem 215:

I cannot think of a really great way of doing this other than the direct way. The direct way is observing that we have infinitely many terms but only two unknown values to solve for. So, it will be sufficient to just take the first couple terms ($n=1$ and $n=2$) and solve for A and B in the resulting system of two equations in two unknowns. For $n = 1$, we have $1 = A(1+\sqrt{5})/2 + B(1-\sqrt{5})/2$, so that $2 = A+B + (A-B)\sqrt{5}$ (which I write in this fashion in the hopes that it might simplify my arithmetic a little). For $n = 2$, we have $1 = A(1+2\sqrt{5}+5)/4 + B(1-2\sqrt{5}+5)/4$, so that $4 = 6(A+B) + (A-B)2\sqrt{5}$ (again, I am hoping that my arithmetic might be simplified writing it this way). If we multiply the first equation by two we see that $4 = 2(A+B) + 2(A-B)\sqrt{5}$. But our second equation is also equal to 4, so that $2(A+B) + 2(A-B)\sqrt{5} = 6(A+B) + (A-B)2\sqrt{5}$. Subtracting $2(A-B)\sqrt{5}$ from both sides, we have $2(A+B) = 6(A+B)$, or $4(A+B) = 0$ which means that $A+B=0$ or that $A = -B$. (Now that was quite a simplification of arithmetic. The reason to think something like this might happen is because of the $1+\sqrt{5}$ and the $1-\sqrt{5}$ -- it seems as though some of the terms must cancel each other out in the arithmetic especially if it all adds up to some integer.) Using the first equation substituting $-B$ in for A , we get $2 = (-B+B) + (-B-B)\sqrt{5} = -2B\sqrt{5}$. And so, $B = -1/\sqrt{5}$ and $A = -B = 1/\sqrt{5}$.

[Top](#) [Education Main Page](#)

Problem 217:

We have (*minor sixth*) $\times$ (*major third*) = (*octave*), so that (*minor sixth*) $\times \frac{5}{4} = 2$. Thus, (*minor sixth*) $= 2 \div \frac{5}{4} = \frac{8}{5}$.

We have (*major sixth*) $\times$ (*minor third*) = (*octave*), so that (*major sixth*) $\times \frac{6}{5} = 2$. Thus, (*major sixth*) $= 2 \div \frac{6}{5} = \frac{10}{6} = \frac{5}{3}$.

Problem 218:

They are all multiplied by a constant number that increases the frequencies if the tone is higher (I believe it's 45 divided by 33 $1/3$?). I'm not sure why we should know this except for what he just said in the previous paragraph.

[Top](#) [Education Main Page](#)

Problem 220:

Basically this question amounts to whether or not the numbers a , $3a/2$, and $2a$ (for some a not equal to zero) can all be in the same geometric progression together which is in turn the same as asking if 1, $3/2$, and 2 can all be in the same progression. So, if q is the common ratio, then $q^n = 3/2$ for some positive integer n and $(3/2) \cdot (q^m) = 2$ for some other positive integer m . Thus, $q^n = 3/2$ and $q^m = 4/3$. Using the same trick as in problem 205, we have that $(3/2)^m = (4/3)^n$ or $3^{n+m} = (4^n) \cdot (2^m) = 2^{2n+m}$. In this case, we actually do know that n and m are positive integers and the only way the equality could possibly hold is if $n = m = 0$. Thus there is no geometric progression with 1, $3/2$, and 2. (In short, to answer Gelfand's question, then, no -- there's no way to get true fifths.)

[Top](#) [Education Main Page](#)

Problem 221:

Well, what do you think?

[Top](#) [Education Main Page](#)

Problem 222:

Achilles racing the turtle corresponds to a geometric progression with common factor $q=1/10$. If we extend this series backwards to $n=-1$, we get that $q^{(-1)} = 10$. That is if the turtle goes one tenth as far to the front of Achilles as Achilles does, then it should go ten times as far to the rear of Achilles as Achilles does. Thus, if Achilles steps backwards $1/9$ meters, the turtle must go backwards $10/9$ meters. Since the turtle starts 1 meter ahead of Achilles, it would reach a position $10/9 - 1 = 1/9$ meters behind Achilles' initial position in so doing. And that is exactly where Achilles would be, as well.

[Top](#) [Education Main Page](#)

Problem 224:

(a) $0 = (x^2) - 4 = (x^2) - (2^2) = (x-2)(x+2)$, so that $x-2=0$ or $x+2=0$ which implies that $x = 2$ or $x = -2$.

(b) $0 = (x^2) + 2 = (x^2) - \{[\sqrt{-1}]\sqrt{2}\}^2 = [x - \sqrt{-1}]\sqrt{2}][x + \sqrt{-1}]\sqrt{2}]$, so that $x = \sqrt{-1}\sqrt{2}$ or $x = -\sqrt{-1}\sqrt{2}$. (Recall the discussion in Factoring.)

(c) $0 = (x^2) - 2x + 1 = (x-1)^2$, so that $x = 1$.

(d) $9 = (x^2) - 2x + 1 = (x-1)^2$. Since $3^2 = 9$ and $(-3)^2 = 9$ (and these are the only ways to square a number to get 9), $x-1 = 3$ or $x-1 = -3$ which implies that $x = 1+3 = 4$ or $x = 1-3 = -2$.

(e) $0 = (x^2) - 2x - 8 = (x^2) - 2x - (2 \cdot 4) = (x^2) + (2-4)x + (2) \cdot (-4) = (x+2)(x-4)$, so that $x = -2$ or $x = 4$.

(f) $0 = (x^2) - 2x - 3 = (x^2) - 2x - (1 \cdot 3) = (x^2) + (1-3)x + (1) \cdot (-3) = (x+1)(x-3)$, so that $x = -1$ or $x = 3$.

(g) $0 = (x^2) - 5x + 6 = (x^2) - 5x + (2 \cdot 3) = (x^2) + (-2x) + (-3x) + (-2) \cdot (-3) = (x-2)(x-3)$, so that $x = 2$ or $x = 3$.

(h) $0 = (x^2) - x - 2 = (x^2) - x - (1 \cdot 2) = (x^2) + (1-2)x + (1) \cdot (-2) = (x+1)(x-2)$, so that $x = -1$ or $x = 2$.

[Top](#) [Education Main Page](#)

Problem 225:

If both a and b are zero, then we are just left with $c=0$. If c is zero then any x will satisfy the equation and otherwise no x will.

[Top](#) [Education Main Page](#)

Problem 229:

Using a similar approach to the preceding problem, $\sqrt{a}/\sqrt{b}$ is a positive number since both $\sqrt{a}$ and $\sqrt{b}$ are positive numbers. Furthermore, $[\sqrt{a}/\sqrt{b}]^2 = [\sqrt{a}]^2/[\sqrt{b}]^2 = a/b$. Thus, $\sqrt{a}/\sqrt{b}$ is a positive number whose square is a/b or $\sqrt{a}/\sqrt{b} = \sqrt{a/b}$.

[Top](#) [Education Main Page](#)

Problem 231:

$$(a)[2-\sqrt{3}][2+\sqrt{3}] = (2^2 - [\sqrt{3}]^2) = 4 - 3 = 1.$$

$$(b)[\sqrt{7}+\sqrt{5}][\sqrt{7}-\sqrt{5}] = [\sqrt{7}]^2 - [\sqrt{5}]^2 = 7 - 5 = 2.$$

[Top](#) [Education Main Page](#)

Problem 232:

$[\sqrt{1001}-\sqrt{1000}][\sqrt{1001}+\sqrt{1000}] = 1001-1000 = 1$. $[\sqrt{1001}+\sqrt{1000}]/10 > [\sqrt{1000}+\sqrt{1000}]/10 = 2*\sqrt{10^3}/10 = 2*10*\sqrt{10}/10 = 2*\sqrt{10} > 2 > 1 = [\sqrt{1001}-\sqrt{1000}][\sqrt{1001}+\sqrt{1000}]$. Thus, dividing by $[\sqrt{1001}+\sqrt{1000}]$, we see that $1/10 > [\sqrt{1001}-\sqrt{1000}]$.

[Top](#) [Education Main Page](#)

Problem 236:

$(x^2)+2x-8 = (x^2)+2x+1-1-8 = (x+2)^2 - 9$. Thus, $(x+1)^2=9$ implies that $x+1=3$ or $x+1=-3$. Thus, $x=2$ or $x=-4$.

[Top](#) [Education Main Page](#)

Problem 239:

The generalization would be to set beta (which I will denote with an italicized b) equal to alpha (which I will denote with an italicized a), to get $2a=-p$ and $a^2=q$. In the first proof is $a=b$, then $D=0$ and $a=b=-p/2$ and the rest of the calculations of the first proof follow. In the second proof, if $P(x)$ has only one root a , then $(x-a)$ can be factored out *twice* since the remaining factor after factoring it out once must be a linear polynomial whose root is a which means it must also be $(x-a)$. So, you can use both methods to prove it in the degenerate case.

[Top](#) [Education Main Page](#)

Problem 240:

We shall use the second approach. We can factor $P(x) = (x-a)(x-b)(x-g)R(x)$ since a , b , and g are roots. Furthermore since $(x-a)(x-b)(x-g)$ is a third degree polynomial with leading term x^3 (the same as given for $P(x)$), $R(x)$ must be equal to one. Then, $P(x) = (x^3)-(a+b+g)(x^2)+(ab+ag+bg)x-abg$. So, equating coefficients, we get the formulas in the book.

Problem 243:

First of all, let me just say that if your child doesn't get this one on their own (or if you do not), don't be too upset. However, there is a method to the madness of hat will become an *explosion* of formulas. And, it's pretty interesting, so I am going to spend a long time writing all this up. Personally, I would give my child the problem to work on for a few days. Then, when they don't get it, give them all the key ideas (which I will discuss below) and let them work on it. Then, work through it with them.

So, in terms of solving this problem , we would tend to imagine that of course there must be some way to write the roots in terms of the coefficients because we know from our math history that the cubic is "solvable". In other words, there's some cubic formula out there like the quadratic formula. Even if we are completely ignorant of this history, it seems somewhat natural to assume that a polynomials roots can be expressed in terms of its coefficients, and in any case, that turns out to be true for the cubic. So, with that in mind, we might approach this problem with the sort of relentlessness of one who *knows* it can be solved. But, it turns out, that if one knows some key facts, what might at first appear to be a tangled mess of formulas turns out to be quite manageable. (Though, this problem still requires an awful lot of just tedious work. It's value is less in how instructive it is as a pardigmatic problem but more in just its value as a fact. In other words, this is probably another "real" problem.)

Key

Formulas/Theorems:

Vieta's

Theorem:

It may seem somewhat trivial since we derived it pretty nonchalantly and in a straight forward fashion in problem 240. But, it turns out, that from the standpoint of mechanical manipulation, it gives us all the key pieces we need to translate roots into coefficients. It gives us an expresion where each root appears all by itself perhaps added to other roots, an expression that has each pairwise product of roots, and an expression where all the roots are multiplied together. So, while it might seem that the equations are really showing you the coefficients in terms of the roots, what they actually give us is three key expressions of the roots in terms of nothing but the coefficients! We will see just how this works in a moment, but for now let me just refresh our memory with the formulas. If a, b, and c are the roots of a cubic $P(x) = x^3 + px^2 + qx + r$, then

$$\begin{array}{lcl} a+b+c & = & -p \\ ab+ac+bc & = & q \\ abc & = & -r \end{array}$$

The Sum of Squared Terms:

Now you might not be able to appreciate the value of Vieta's Theorem and how on a practical level it really is telling us something about the roots in terms of the coefficients as opposed to something about the coefficients in terms of the roots if you do not have a lot of experience

with this and the next key fact. What is the sum of three squared terms: $a^2 + b^2 + c^2$? The question, itself, might not make sense, but consider the square of the sum:

$$\begin{aligned}
 &(a+b+c)^2 \\
 &= \\
 &= \quad \quad \quad a(a+b+c) \quad \quad + \quad \quad \quad b(a+b+c) \quad \quad + \quad \quad \quad c(a+b+c) \\
 &= \quad a^2 \quad + \quad ab \quad + \quad ac \quad + \quad ba \quad + \quad b^2 \quad + \quad bc \quad + \quad ca \quad + \quad cb \quad + \quad c^2 \\
 &= \quad a^2 \quad + \quad b^2 \quad + \quad c^2 \quad + \quad 2(ab \quad + \quad ac \quad + \quad bc)
 \end{aligned}$$

Do you see anything you recognize from Vieta's Theorem? In fact, if I rearrange, now, these expression I can get the following formula (the key fact we shall later use):

$$a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ac)$$

In other words, I can write the sum of squared terms in terms of an expression involving only combinations of terms to the first power (but in which part of the expression may well be squared). And, moreover, with Vieta's Theorem we can directly translate the sum of squared roots into an expression in terms of only the coefficients of the polynomial because Vieta's Theorem gives us such translations for very way of combining the roots with one another.

The Sum of Cubed Terms:

We easily enough derived a formula for the sum of squared that gave us an expression involving nothing bu the terms raised to the first power. We will need to do the same for the sum of cubed terms. (And, we need only do it for three terms, by the way, because there are only three roots.) Recall Problem 122(d) in which we factored:

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)*[(a^2 + b^2 + c^2) - (ab + bc + ac)]$$

Now, using the above key fact for the sum of squared terms, we can rearrange this equation to get us the sum of cubed terms in terms of combinations of the terms to single powers:

$$\begin{aligned}
 &a^3 \quad \quad \quad + \quad \quad \quad b^3 \quad \quad \quad + \quad \quad \quad c^3 \\
 &= \quad (a+b+c)*[(a^2 \quad + \quad b^2 \quad + \quad c^2) \quad - \quad (ab+bc+ac)] \quad + \quad 3abc \\
 &= \quad (a+b+c)*[(a+b+c)^2 \quad - \quad 2*(ab+bc+ac) \quad - \quad (ab+bc+ac)] \quad + \quad 3abc \\
 &= \quad (a+b+c)*[(a+b+c)^2 \quad - \quad 3*(ab+bc+ac)] \quad + \quad 3abc
 \end{aligned}$$

And, you can see how you will use every formula in Vieta's Theorem for Cubics here. Vieta's innocuous little theorem which seemingly just gives us some albeit straightforward but unuseable result is actually much more than that -- it is just *precisely* what we need (and nothing we don't)!

Finally the Solution:

I will use a, b, and c as my roots (rather than x subscript 1, x subscript 2 and x subscript 3). And for the coefficients, I will use p, q, and r as they appear in the book. From our key facts, we have the following two formulas:

$$a^3 + b^3 + c^3$$

$$\begin{aligned}
&= (a+b+c)*[(a+b+c)^2 - 3(ab+bc+ac)] + 3abc \\
&= (-p)[(-p)^2 - 3q] + 3(-r) \\
&= -p^3 + 3pq - 3r
\end{aligned}$$

and

$$\begin{aligned}
&a^2 + b^2 + c^2 \\
&= (a+b+c)^2 - 2(ab+bc+ac) \\
&= (-p)^2 - 2q \\
&= p^2 - 2q
\end{aligned}$$

Now, we just have to let the problem explode into a big mess and multiply out the product of squared differences:

$$\begin{aligned}
&[(a-b)^2][(b-c)^2][(a-c)^2] \\
&= (a^2 - 2ab + b^2)(b^2 - 2bc + c^2)(a^2 - 2ac + c^2) \\
&= [(a^2)(b^2) - 2(a^2)bc + (a^2)(c^2) - 2a(b^3) + 4a(b^2)c - 2ab(c^2) \\
&\quad + (b^4) - 2(b^3)c + (b^2)(c^2)](a^2 - 2ac + c^2) \\
&= (a^2)*[(a^2)(b^2) - 2(a^2)bc + (a^2)(c^2) - 2a(b^3) + 4a(b^2)c - 2ab(c^2) \\
&\quad + (b^4) - 2(b^3)c + (b^2)(c^2)] \\
&\quad - 2ac*[(a^2)(b^2) - 2(a^2)bc + (a^2)(c^2) - 2a(b^3) + 4a(b^2)c - 2ab(c^2) \\
&\quad + (b^4) - 2(b^3)c + (b^2)(c^2)] \\
&\quad + (c^2)*[(a^2)(b^2) - 2(a^2)bc + (a^2)(c^2) - 2a(b^3) + 4a(b^2)c - 2ab(c^2) \\
&\quad + (b^4) - 2(b^3)c + (b^2)(c^2)] \\
&= (a^4)(b^2) - 2(a^4)bc + (a^4)(c^2) - 2(a^3)(b^3) + 4(a^3)(b^2)c - 2(a^3)b(c^2) \\
&\quad + (a^2)(b^4) - 2(a^2)(b^3)c + (a^2)(b^2)(c^2) - 2(a^3)(b^2)c + 4(a^3)b(c^2) \\
&\quad - 2(a^3)(c^3) + 4(a^2)(b^3)c - 8(a^2)(b^2)(c^2) + 4(a^2)b(c^3) - 2a(b^4)c \\
&\quad + 4a(b^3)(c^2) - 2a(b^2)(c^3) + (c^2)(a^2)(b^2) - 2(a^2)b(c^3) + (a^2)(c^4) \\
&\quad - 2a(b^3)(c^2) + 4a(b^2)(c^3) - 2ab(c^4) + (c^2)(b^4) - 2(b^3)(c^3) + (b^2)(c^4)
\end{aligned}$$

Now, *that* is an explosion of crap! To manage such a large amount of crap like this, You will have to decide on an algorithm for going through it term by term to accomplishing the distributing of a product through a sum or to regroup and so on. What I will do first, now that I have it all multiplied out is use the last formula in Vieta's Theorem ($abc=-r$) to help reduce the crap as much as I can and otherwise combine like terms and simplify. So, we proceed to get:

$$\begin{aligned}
&(a^4)(b^2) - 2(a^4)bc + (a^4)(c^2) - 2(a^3)(b^3) + 4(a^3)(b^2)c - 2(a^3)b(c^2) \\
&\quad + (a^2)(b^4) - 2(a^2)(b^3)c + (a^2)(b^2)(c^2) - 2(a^3)(b^2)c + 4(a^3)b(c^2) \\
&\quad - 2(a^3)(c^3) + 4(a^2)(b^3)c - 8(a^2)(b^2)(c^2) + 4(a^2)b(c^3) - 2a(b^4)c \\
&\quad + 4a(b^3)(c^2) - 2a(b^2)(c^3) + (c^2)(a^2)(b^2) - 2(a^2)b(c^3) + (a^2)(c^4)
\end{aligned}$$

$$-2a(b^3)(c^2)+4a(b^2)(c^3)-2ab(c^4)+(c^2)(b^4)-2(b^3)(c^3)+(b^2)(c^4)$$

$$= (a^4)(b^2)-2(a^3)(-r)+(a^4)(c^2)-2(a^3)(b^3)+4(a^2)b(-r)-2(a^2)c(-r) \\ + (a^2)(b^4)-2a(b^2)(-r)+((-r)^2)-2(a^2)b(-r)+4(a^2)c(-r)-2(a^3)(c^3) \\ +4a(b^2)(-r)-8((-r)^2)+4a(c^2)(-r)-2(b^3)(-r)+4(b^2)c(-r)-2b(c^2)(-r) \\ +((-r)^2)-2a(c^2)(-r)+(a^2)(c^4)-2(b^2)c(-r)+4b(c^2)(-r)-2(c^3)(-r) \\ +(c^2)(b^4)-2(b^3)(c^3)+(b^2)(c^4)$$

$$= (a^4)(b^2)+2(a^3)r+(a^4)(c^2)-2(a^3)(b^3)-4(a^2)br+2(a^2)cr \\ + (a^2)(b^4)+2a(b^2)r+(r^2)+2(a^2)br-4(a^2)cr-2(a^3)(c^3) \\ -4a(b^2)r-8(r^2)-4a(c^2)r+2(b^3)r-4(b^2)cr+2b(c^2)r \\ +(r^2)+2a(c^2)r+(a^2)(c^4)+2(b^2)cr-4b(c^2)r+2(c^3)r \\ +(c^2)(b^4)-2(b^3)(c^3)+(b^2)(c^4)$$

Just substituting in $(-r)$ for abc really cleared up some of the clutter. I went through term by term to see if I could put a $(-r)$ in for an (abc) and did so if I could or otherwise went on to the next term. Don't try to "do it all at once" or do more than one small step at a time when confronted with such a mess. You will end up making some careless error and rapidly become hopelessly lost. Now, systematically by copying each like term down next to one another as you go through the expression term-by-term, combine like terms to get a polynomial in r with the clutter of a 's, b 's, and c 's as the coefficients of various powers of r to get:

$$= (a^4)(b^2)+2(a^3)r+(a^4)(c^2)-2(a^3)(b^3)-4(a^2)br+2(a^2)cr \\ + (a^2)(b^4)+2a(b^2)r+(r^2)+2(a^2)br-4(a^2)cr-2(a^3)(c^3) \\ -4a(b^2)r-8(r^2)-4a(c^2)r+2(b^3)r-4(b^2)cr+2b(c^2)r \\ +(r^2)+2a(c^2)r+(a^2)(c^4)+2(b^2)cr-4b(c^2)r+2(c^3)r \\ +(c^2)(b^4)-2(b^3)(c^3)+(b^2)(c^4)$$

$$= +(r^2)-8(r^2)+(r^2) \\ +2(a^3)r-4(a^2)br+2(a^2)cr+2a(b^2)r+2(a^2)br-4(a^2)cr-4a(b^2)r-4a(c^2)r \\ +2(b^3)r-4(b^2)cr+2b(c^2)r+2a(c^2)r+2(b^2)cr-4b(c^2)r+2(c^3)r \\ +(a^4)(b^2)+(a^4)(c^2)-2(a^3)(b^3)+(a^2)(b^4)-2(a^3)(c^3)+(a^2)(c^4) \\ +(c^2)(b^4)-2(b^3)(c^3)+(b^2)(c^4)$$

$$= -6(r^2) \\ +[2(a^3)-4(a^2)b+2(a^2)c+2a(b^2)+2(a^2)b-4(a^2)c-4a(b^2)-4a(c^2) \\ +2(b^3)-4(b^2)c+2b(c^2)+2a(c^2)+2(b^2)c-4b(c^2)+2(c^3)]*r \\ +(a^4)(b^2)+(a^4)(c^2)-2(a^3)(b^3)+(a^2)(b^4)-2(a^3)(c^3)+(a^2)(c^4) \\ +(c^2)(b^4)-2(b^3)(c^3)+(b^2)(c^4)$$

$$= -6(r^2) \\ +[2(a^3)+2(b^3)+2(c^3)+2(a^2)c+2a(b^2)+2(a^2)b+2b(c^2)+2a(c^2)+2(b^2)c$$

$$\begin{aligned}
& -4(a^2)c-4a(b^2)-4a(c^2)-4(a^2)b-4(b^2)c-4b(c^2)]*r \\
& +(a^4)(b^2)+(a^4)(c^2)+(a^2)(b^4)+(a^2)(c^4)+(b^2)(c^4)+(c^2)(b^4) \\
& -2(a^3)(b^3)-2(a^3)(c^3)-2(b^3)(c^3)
\end{aligned}$$

In the last step above, I rearrange the pieces to put all the stuff multiplied by 2 together, all the stuff multiplied by four together, and so on. There is no particular reason why I knew to do that - it is just sort of a guess made in an effort to make the expression more organized. Doing so, however, has started us on our next series of manipulations in which we will try to use the two key formulas above for the sum of cubes and the sum of squares to get rid of as many a's, b's, and c's as we can. We will have to do some manipulation along the way to do it though, but instead of it just being some random attempts at trying a pattern here or there, we will specifically try to conjure up as many sums of the following forms: $a^3+b^3+c^3$, $a^2+b^2+c^2$, $a+b+c$, or $ab+ac+bc$ as we can so that we can then substitute the equivalent expressions of p's, q's, and r's.

$$\begin{aligned}
& = -6(r^2) \\
& \quad +[2(a^3)+2(b^3)+2(c^3)+2(a^2)c+2a(b^2)+2(a^2)b+2b(c^2)+2a(c^2)+2(b^2)c \\
& \quad \quad -4(a^2)c-4a(b^2)-4a(c^2)-4(a^2)b-4(b^2)c-4b(c^2)]*r \\
& \quad + (a^4)(b^2)+(a^4)(c^2)+(a^2)(b^4)+(a^2)(c^4)+(b^2)(c^4)+(c^2)(b^4) \\
& \quad \quad -2(a^3)(b^3)-2(a^3)(c^3)-2(b^3)(c^3)
\end{aligned}$$

$$\begin{aligned}
& = -6(r^2) \\
& \quad +\{2[(a^3)+(b^3)+(c^3)]+2(a^2)c+2(a^2)b+2a(b^2)+2(b^2)c+2b(c^2)+2a(c^2) \\
& \quad \quad -4(a^2)b-4(a^2)c-4a(b^2)-4(b^2)c-4a(c^2)-4b(c^2)\}*r \\
& \quad + (a^4)(b^2)+(a^4)(c^2)+(a^2)(b^4)+(c^2)(b^4)+(a^2)(c^4)+(b^2)(c^4) \\
& \quad \quad -2(a^3)(b^3)-2(a^3)(c^3)-2(b^3)(c^3)
\end{aligned}$$

$$\begin{aligned}
& = -6(r^2) \\
& \quad +\{2[-p^3 \quad + \quad 3pq \quad - \quad 3r]+2(a^2)(c+b)+2(b^2)(a+c)+2(c^2)(b+a) \\
& \quad \quad -4(a^2)(b+c)-4(b^2)(a+c)-4(c^2)(a+b)\}*r \\
& \quad + (a^4)[(b^2)+(c^2)]+(b^4)[(a^2)+(c^2)]+(c^4)[(a^2)+(b^2)] \\
& \quad \quad -2[(a^3)(b^3)+(a^3)(c^3)+(b^3)(c^3)]
\end{aligned}$$

$$\begin{aligned}
& = -6(r^2)-2[p^3-3pq+3r]r \\
& \quad +\{2(a^2)(a+b+c-a)+2(b^2)(a+b+c-b)+2(c^2)(a+b+c-c) \\
& \quad \quad -4(a^2)(a+b+c-a)-4(b^2)(a+b+c-b)-4(c^2)(a+b+c-c)\}*r \\
& \quad + (a^4)[(a^2)+(b^2)+(c^2)-(a^2)]+(b^4)[(a^2)+(b^2)+(c^2)-(b^2)] \\
& \quad + (c^4)[(a^2)+(b^2)+(c^2)-(c^2)]-2[(a^3)(b^3)+(a^3)(c^3)+(b^3)(c^3)]
\end{aligned}$$

$$\begin{aligned}
& = -6(r^2)-2[p^3-3pq+3r]r \\
& \quad +\{2(a^2)(-p-a)+2(b^2)(-p-b)+2(c^2)(-p-c) \\
& \quad \quad -4(a^2)(-p-a)-4(b^2)(-p-b)-4(c^2)(-p-c)\}*r \\
& \quad + (a^4)[(p^2-2q)-(a^2)]+(b^4)[(p^2-2q)-(b^2)]
\end{aligned}$$

$$\begin{aligned}
& +(c^4)[(p^2-2q)-(c^2)]-2[(a^3)(b^3)+(a^3)(c^3)+(b^3)(c^3)] \\
= & -6(r^2)-2[p^3-3pq+3r]r \\
& +\{-2(a^2)p-2(a^3)-2(b^2)p-2(b^3)-2(c^2)p-2(c^3) \\
& +4(a^2)p+4(a^3)+4(b^2)p+4(b^3)+4(c^2)p+4(c^3)\}^*r \\
& +(a^4)(p^2-2q)-(a^6)+(b^4)(p^2-2q)-(b^6) \\
& +(c^4)(p^2-2q)-(c^6)-2[(a^3)(b^3)+(a^3)(c^3)+(b^3)(c^3)] \\
= & -6(r^2)-2[p^3-3pq+3r]r \\
& +\{-2(a^2)p-2(b^2)p-2(c^2)p+4(a^2)p+4(b^2)p+4(c^2)p \\
& -2(a^3)-2(b^3)-2(c^3)+4(a^3)+4(b^3)+4(c^3)\}^*r \\
& +(a^4)(p^2-2q)+(b^4)(p^2-2q)+(c^4)(p^2-2q) \\
& -(a^6)-(b^6)-(c^6)-2[(a^3)(b^3)+(a^3)(c^3)+(b^3)(c^3)] \\
= & -6(r^2)-2[p^3-3pq+3r]r \\
& +[2p^*(a^2)+2p^*(b^2)+2p^*(c^2) \\
& +2(a^3)+2(b^3)+2(c^3)]^*r \\
& +[(a^4)+(b^4)+(c^4)](p^2-2q) \\
& -\{(a^6)+(b^6)+(c^6)+2[(a^3)(b^3)+(a^3)(c^3)+(b^3)(c^3)]\} \\
= & -6(r^2)-2[p^3-3pq+3r]r \\
& +\{2p^*[(a^2)+(b^2)+(c^2)]+2^*[(a^3)+(b^3)+(c^3)]\}^*r \\
& +[(a^4)+(b^4)+(c^4)](p^2-2q) \\
& -\{(a^6)+(b^6)+(c^6)+2[(a^3)(b^3)+(a^3)(c^3)+(b^3)(c^3)]\} \\
= & -6(r^2)-2[p^3-3pq+3r]r \\
& +[2p^*(p^2-2q)+2^*(-p^3+3pq-3r)]^*r \\
& +[(a^4)+(b^4)+(c^4)](p^2-2q) \\
& -\{(a^6)+(b^6)+(c^6)+2[(a^3)(b^3)+(a^3)(c^3)+(b^3)(c^3)]\} \\
= & -6(r^2)-4(p^3-3pq+3r)r+2p^*(p^2-2q)^*r \\
& +[(a^4)+(b^4)+(c^4)](p^2-2q) \\
& -\{(a^6)+(b^6)+(c^6)+2[(a^3)(b^3)+(a^3)(c^3)+(b^3)(c^3)]\}
\end{aligned}$$

Argh! So close! That last chunk just didn't go as we expected! We have the sum of fourth powers on the one hand and then there is the mess of powers of six that follows it! And, all of that is followed up with *mixed* powers of three. Actually, it is not as bad as it seems. The powers of four are really just squared terms squared. That is, we can use the same approach we did in deriving the second key fact twice to handle this sum of 4th powers (but we have to be a little clever about it):

$$\begin{aligned}
& (a^4)+(b^4)+(c^4) \\
= & \\
= & (a^2)^2+(b^2)^2+(c^2)^2 \\
& ((a^2)+(b^2)+(c^2))^2-2[(a^2)(b^2)+(b^2)(c^2)+(a^2)(c^2)]
\end{aligned}$$

$$\begin{aligned}
&= ((a^2)+(b^2)+(c^2))^2-2[(ab)^2+(bc)^2+(ac)^2] \\
&= (p^2-2q)^2-2[(ab)^2+(bc)^2+(ac)^2] \\
&= (p^4-4qp^2+4q^2)-2\{[(ab)+(bc)+(ac)]^2-2(abbc+bcac+abac)\} \\
&= (p^4-4qp^2+4q^2)-2[q-2(br+cr+ar)] \\
&= (p^4-4qp^2+4q^2)-2q^2+4(a+b+c)(-r) \\
&= p^4-4qp^2+4q^2-2q^2+4(-p)(-r) \\
&= p^4-4qp^2+2q^2+4pr
\end{aligned}$$

We used essentially the same trick that we did way up above to get the second key fact which was the formula for the sum of squared terms, but we also had some clutter left behind that we could use the third formula of Vieta's Theorem to clean up.

So now we are left with the expression:

$$\begin{aligned}
&-6(r^2)-4(p^3-3pq+3r)r+2p(p^2-2q)r \\
&\quad +[(a^4)+(b^4)+(c^4)](p^2-2q)-\{(a^6)+(b^6)+(c^6) \\
&\quad +2[(a^3)(b^3)+(a^3)(c^3)+(b^3)(c^3)]\} \\
&= -6(r^2)-4(p^3-3pq+3r)r+2p(p^2-2q)r+(p^4-4qp^2+2q^2+4pr)(p^2-2q) \\
&\quad -\{(a^6)+(b^6)+(c^6)+2[(a^3)(b^3)+(a^3)(c^3)+(b^3)(c^3)]\}
\end{aligned}$$

Now the final bits of a, b, and c might seem pretty daunting looking, but recall that

$$\begin{aligned}
&(x+y+z)^2 \\
&= (x^2)+(y^2)+(z^2)+2(xy+yz+xz)
\end{aligned}$$

So that if $x=a^3$, $y=b^3$, and $z=c^3$, we would have that

$$\begin{aligned}
&[(a^3)+(b^3)+(c^3)]^2 \\
&= ((a^3)^2)+((b^3)^2)+((c^3)^2)+2((a^3)(b^3)+(b^3)(c^3)+(a^3)(c^3))
\end{aligned}$$

which is *precisely* our expression that we have yet to convert to p's, q's, and r's. Therefore, we may (finally) write out the product of the squared differences of the roots as (drumroll please!)

$$\begin{aligned}
&-6(r^2)-4(p^3-3pq+3r)r+2p(p^2-2q)r+(p^4-4qp^2+2q^2+4pr)(p^2-2q) \\
&\quad -\{(a^6)+(b^6)+(c^6)+2[(a^3)(b^3)+(a^3)(c^3)+(b^3)(c^3)]\} \\
&= -6(r^2)-4(p^3-3pq+3r)r+2p(p^2-2q)r \\
&\quad + (p^4-4qp^2+2q^2+4pr)(p^2-2q)-[(a^3)+(b^3)+(c^3)]^2 \\
&= -6(r^2)-4(p^3-3pq+3r)r+2p(p^2-2q)r \\
&\quad + (p^4-4qp^2+2q^2+4pr)(p^2-2q)-(-p^3+3pq-3r)^2 \\
&= -6(r^2)-4(p^3-3pq+3r)r+2p(p^2-2q)r \\
&\quad + (p^4-4qp^2+2q^2+4pr)(p^2-2q)-[-(p^3-3pq+3r)]^2 \\
&= -6(r^2)-4(p^3-3pq+3r)r+2p(p^2-2q)r \\
&\quad + (p^4-4qp^2+2q^2+4pr)(p^2-2q)-(p^3-3pq+3r)^2
\end{aligned}$$

I'm afraid we still are not done yet. To finally put this problem to bed, we must multiply out this still ugly expression and simplify it (combining like terms and all that). So, we have

$$-6(r^2)-4(p^3-3pq+3r)r+2p(p^2-2q)r$$

$$\begin{aligned}
& +(p^4-4qp^2+2q^2+4pr)(p^2-2q)-(p^3-3pq+3r)^2 \\
= & -6r^2-4(p^3)r+12pqr-12r^2+2p^3r-4pqr+p^6-4qp^4+2(p^2)(q^2)+4rp^3 \\
& -2qp^4+8(q^2)(p^2)-4q^3-8pqr-(p^3-3pq+3r)^2 \\
= & -18r^2+2(p^3)r+p^6-6qp^4+10(p^2)(q^2)-4q^3 \\
& -[p^6+9(p^2)(q^2)+9r^2+2(-3qp^4+3rp^3-9pqr)] \\
= & -27r^2-4q^3-4rp^3+(p^2)(q^2)+18pqr
\end{aligned}$$

[Top](#) [Education Main Page](#)

Problem 244:

I will use a and b and c and d as the roots of the two polynomials, and so similar to the preceding problem, let's just multiply the expression out and see what we get:

$$\begin{aligned}
& (a-c)(b-c)(a-d)(b-d) \\
= & [ab-bc-ac+(c^2)][ab-bd-ad+(d^2)] \\
= & [(ab)^2]-a(b^2)d-(a^2)bd+ab(d^2)-a(b^2)c+(b^2)cd+abcd-bc(d^2)-(a^2)bc+abcd \\
& +(a^2)cd-ac(d^2)+ab(c^2)-b(c^2)d-a(c^2)d+[(cd)^2]
\end{aligned}$$

Using the Vieta's Theorem we can substitute $ab=q$ and $cd=s$ to get

$$\begin{aligned}
& [(ab)^2]-a(b^2)d-(a^2)bd+ab(d^2)-a(b^2)c+(b^2)cd+abcd-bc(d^2)-(a^2)bc+abcd \\
& +(a^2)cd-ac(d^2)+ab(c^2)-b(c^2)d-a(c^2)d+[(cd)^2] \\
= & (q^2)-qbd-aqd+q(d^2)-qbc+(b^2)s+2qs-bsd-aqc+(a^2)s-asd+q(c^2)-bcs-acs+(s^2) \\
= & (q^2)-qb(c+d)-aq(c+d)+q(d^2)+(b^2)s+2qs+(a^2)s-(a+b)sd+q(c^2)-(a+b)cs+(s^2)
\end{aligned}$$

Using Vieta's Theorem again, we can make the substitutions $a+b=-p$ and $c+d=-r$:

$$\begin{aligned}
& (q^2)-qb(c+d)-aq(c+d)+q(d^2)+(b^2)s+2qs+(a^2)s-(a+b)sd+q(c^2)-(a+b)cs+(s^2) \\
= & (q^2)+qbr+aqr+q(d^2)+(b^2)s+2qs+(a^2)s+psd+q(c^2)+pcs+(s^2) \\
= & (q^2)+(a+b)qr+(a^2+b^2)s+2qs+ps(c+d)+q(c^2+d^2)+(s^2) \\
= & (q^2)-pqr+(a^2+b^2)s+2qs-psr+q(c^2+d^2)+(s^2) \\
= & (q^2)+2qs+(s^2)-pqr-psr+(a^2+2ab+b^2-2ab)s+q(c^2+2cd+d^2-2cd) \\
= & (q^2)+2qs+(s^2)-pqr-psr+[(a+b)^2-2q]s+q[(c+d)^2-2s] \\
= & (q^2)+2qs+(s^2)-pqr-psr+[(-p)^2-2q]s+q[(-r)^2-2s] \\
= & (q^2)+2qs+(s^2)-pqr-psr+[p^2-2q]s+q[r^2-2s] \\
= & (q^2)+2qs+(s^2)-pqr-psr+sp^2-2qs+qr^2-2qs \\
= & (q^2)-2qs+(s^2)+sp^2-psr+qr^2-pqr \\
= & (q-s)^2+sp(p-r)+qr(r-p)
\end{aligned}$$

[Top](#) [Education Main Page](#)

Problem 245:

Using the hint, try the product

$$\begin{aligned}
 & [x-(4-\sqrt{7})][x-(4+\sqrt{7})] \\
 &= x^2 - (4-\sqrt{7})x - (4+\sqrt{7})x + (4-\sqrt{7})(4+\sqrt{7}) \\
 &= x^2 - [(4-\sqrt{7})+(4+\sqrt{7})]x + (4^2-4\sqrt{7}+4\sqrt{7}-7) \\
 &= x^2 - 8x + 9
 \end{aligned}$$

These roots are conjugates of each other. So, their product is an integer and their sum is an integer, as well. Of course, if we multiply two linear polynomials together we get a quadratic.

[Top](#)

Problem 246:

(a) Let the roots be denoted a and b . Then, recall that

$$(a+b)^2 = a^2 + 2ab + b^2$$

Subtracting $2ab$ from both sides and applying Vieta's Theorem, we see that the sum of squares is

$$\begin{aligned}
 & a^2 + b^2 \\
 &= (a+b)^2 - 2ab \\
 &= (-p)^2 - 2q \\
 &= p^2 - 2q
 \end{aligned}$$

(I've derived this one before several times.) Since p and q are integers then p^2-2q is an integer, as well. Thus the sum of squared roots is an integer.

(b) We know that the sum of squared roots is an integer. We also know by Vieta's Theorem that the sum of the roots $a+b=-p$ is an integer since p is. The product of two integers is an integer so that

$$\begin{aligned}
 & (a^2+b^2)(a+b) \\
 &= (a^3+a^2b + ab^2+b^3) \\
 &= (a^3+b^3) + a(ab)+b(ab) \\
 &= (a^3+b^3) + (a+b)(ab) \\
 &= (a^3+b^3) + (-p)(q) \\
 &= (a^3+b^3) - pq
 \end{aligned}$$

is an integer. Finally, pq is an integer because both p and q are, and so adding an integer to an integer is an integer, so that

$$\begin{aligned}
 & (a^3+b^3-pq)+pq \\
 &= a^3+b^3
 \end{aligned}$$

is an integer.

- (c) You really cannot prove this without using the Principle of Mathematical Induction. Nevertheless, let us take a look at why this is true. Consider the product as in (b) only using n instead of three:

$$\begin{aligned}
 & (a^{(n-1)} + b^{(n-1)})(a+b) \\
 & = \\
 & = \\
 & = \\
 & = \\
 & \qquad \qquad \qquad a^n + a^{(n-1)}b + ab^{(n-1)} + b^n \\
 & \qquad \qquad \qquad (a^n + b^n) + a^{(n-2)}(ab) + b^{(n-2)}(ab) \\
 & \qquad \qquad \qquad (a^n + b^n) + a^{(n-2)}b + b^{(n-2)}a \\
 & \qquad \qquad \qquad (a^n + b^n) + (a^{(n-2)} + b^{(n-2)})(a+b)
 \end{aligned}$$

Subtracting $(a^{(n-2)} + b^{(n-2)})q$ from both sides we get

$$\begin{aligned}
 & a^n + b^n \\
 & = \qquad \qquad \qquad (a^{(n-1)} + b^{(n-1)})(a+b) \qquad \qquad \qquad - \qquad \qquad \qquad (a^{(n-2)} + b^{(n-2)})q \\
 & = \qquad \qquad \qquad (a^{(n-1)} + b^{(n-1)})(-p) \qquad \qquad \qquad - \qquad \qquad \qquad (a^{(n-2)} + b^{(n-2)})q \\
 & = -p(a^{(n-1)} + b^{(n-1)}) - q(a^{(n-2)} + b^{(n-2)})
 \end{aligned}$$

If $(a^{(n-1)} + b^{(n-1)})$ and $(a^{(n-2)} + b^{(n-2)})$ are both integers, then this last expression will be an integer because p and q are, and so the sum of the roots raised to the n^{th} power will be an integer. Since we have already show that for $n=2$ and $n=3$ the sum of the n^{th} powers of the roots is an integer, we would be able to say that by the Principle of Mathematical Induction, we have proven that the sum of n^{th} powers of the roots is an integer. (Intuitively, we have shown the "base step" of $n=2$ and $n=3$. We have also shown that if we got integers using powers of $n-1$ and $n-2$, then the sum of the roots raised to the n^{th} power will be an integer. These two facts show that using $n=4$ will produce an integer. Then, since $n=3$ and $n=4$ produced integers, $n=5$ will have to produce an integer. Since $n=4$ and $n=5$ produced integers, $n=6$ will have to as well. And so on for all n .)

[Top](#)

Problem 247:

- (a) Just square it out:

$$\begin{aligned}
 & (a + (\sqrt{2})b)^2 \\
 & = \\
 & = \\
 & = k + l(\sqrt{2}) \\
 & \qquad \qquad \qquad a^2 + 2(\sqrt{2})ab + 2b^2 \\
 & \qquad \qquad \qquad a^2 + 2b^2 + 2ab(\sqrt{2})
 \end{aligned}$$

where $k = a^2 + 2b^2$ and $l = 2ab$ are both integers since a and b are integers.

- (b) Again, you really need to use the Principle of Mathematical Induction here to really do the rigorous proof. Informally, though, suppose that $(a + b(\sqrt{2}))^{(n-1)}$ was of the form $k + l(\sqrt{2})$ for some integers k and l . Then,

$$\begin{aligned}
& (a+b(\sqrt{2}))^n \\
& = \\
& = \\
& = ka+2lb+(a+bk)(\sqrt{2})
\end{aligned}
\qquad
\begin{aligned}
& [a+b(\sqrt{2})]^{(n-1)}[a+b(\sqrt{2})] \\
& [k+l(\sqrt{2})][a+b(\sqrt{2})]
\end{aligned}$$

(I did that last multiplication a bit quickly, skipping a few steps.) Since, k , l , a , b , and 2 are all integers, any combination of adding, subtracting or multiplying them together is. Thus, $ka+2lb$ is an integer and $(a+bk)$ is an integer so that $(a+b(\sqrt{2}))^n$ has the desired form. So, we have shown that raising the expression to the 2nd power produces the desired form and that if raising the expression to the $n-1$ power produces the desired form then raising it to the n^{th} power will also. That is, our proposition (that the expression raised to a power is of a certain form) is true for $n=2$ and if it is true for $n=k$, then it is true for $n=k+1$. Then, it being true for $n=2$ forces it to be true for $n=2+1=3$. And, it being true for $n=3$ forces it to be true for $n=3+1=4$. And so on for all n .

(c) Just substituting “ $-b$ ” everywhere we see a “ b ” in the above, we see that the same holds for $(a-b(\sqrt{2}))^n$.

(d) *Solved in the book.*

[Top](#)

Problem 250:

$$2x^2+2x+1/2 = 2(x^2+x+1/4)$$

Following Gelfand’s lead in problem 249, we can use the formula from section 51 and observe that the equation

$$x^2+x+1/4 = 0$$

has two solutions:

$$x_{1,2} = -1/2 \pm \sqrt{1/4-(1/4)^2} = -1/2 \pm \sqrt{(4-1)/4^2} = -1/2 \pm (\sqrt{3})/4 = [-2 \pm (\sqrt{3})]/4$$

And so,

$$\begin{aligned}
& 2x^2+2x+1/2 \\
& = 2(x^2+x+1/4) \\
& = 2[x - ([-2 + (\sqrt{3})]/4)][x - ([-2 - (\sqrt{3})]/4)] \\
& = 2[x - ([-2 + (\sqrt{3})]/4)][x - ([-2 - (\sqrt{3})]/4)] \\
& = 2[x + ([2 - (\sqrt{3})]/4)][x + ([2 + (\sqrt{3})]/4)]
\end{aligned}$$

[Top](#)

Problem 252:

This formula in section 54 is the standard form of the quadratic formula. The reason we don't need to worry about the absolute value is because we actually want both cases of where $2a < 0$ as well as where $2a > 0$. We already are getting both of those by including the " $\pm$ ", so essentially we can just take the square root without worrying about sign in that case.

[Top](#)

Problem 254:

Let us multiply the both the numerator and denominator of standard form of the quadratic formula by the conjugate of the numerator (thereby simply multiplying by 1). To simplify the calculation, I will use a bold " $\pm$ " to signify that when the normal " $\pm$ " is plus then the bold one is minus and conversely when the normal one is minus the bold one is plus. In other words, the " $\pm$ " is a short hand for two different expressions, one plus and one minus, while the bold " $\pm$ " is shorthand for the two conjugate equations.

$$\begin{aligned} & [-b \pm \sqrt{b^2 - 4ac}] / 2a \\ &= [-b \pm \sqrt{b^2 - 4ac}] / 2a \times [-b \pm \sqrt{b^2 - 4ac}] / [-b \pm \sqrt{b^2 - 4ac}] \\ &= ([-b \pm \sqrt{b^2 - 4ac}] \times [-b \pm \sqrt{b^2 - 4ac}]) / (2a \times [-b \pm \sqrt{b^2 - 4ac}]) \\ &= ((-b)^2 - [\sqrt{b^2 - 4ac}]^2) / (2a \times [-b \pm \sqrt{b^2 - 4ac}]) \\ &= [b^2 - (b^2 - 4ac)] / (2a \times [-b \pm \sqrt{b^2 - 4ac}]) \\ &= 4ac / (2a \times [-b \pm \sqrt{b^2 - 4ac}]) \\ &= (4ac/2a) / [-b \pm \sqrt{b^2 - 4ac}] \\ &= 2c / [-b \pm \sqrt{b^2 - 4ac}] \end{aligned}$$

Of course, it doesn't matter in the end if the denominator was originally arranged to be the conjugate of the numerator of the standard form, so we could just drop that and use the regular " $\pm$ ".

[Top](#)

Problem 256:

- (a) The roots of the equation have the form: $x_{1,2} = [1 \pm \sqrt{1+4a}]/2$. Thus, as a increases the roots get further away from zero (and each other). And conversely, as a decreases.
- (b) The roots of the equation have the form: $x_{1,2} = [a \pm \sqrt{a^2 - 4}]/2$. If $-2 < a < 2$ then there are no (real) roots. For values of a such that $|a|$ is just slightly greater than 2, the roots are at their furthest from each other. But, as $|a|$ becomes larger and larger, the roots get closer and closer to one another.

[Top](#)

Problem 259:

First of all, it doesn't seem particularly clear what "shift" or "stretch" really mean in terms of the algebra and the graph. To shift horizontally, one adds a value directly to the x and uses " $x+m$ " instead of " x " in the formula. To shift vertically, one adds a value directly to the y and uses " $y-n$ " instead of " y " in the formula. Note that subtracting a number is really just adding its negative and so on so that I am accomplishing the same thing by subtracting rather than adding in terms of a vertical shift, although the direction of the shift is reversed. I am subtracting here to remain consistent with the book so that the punchline is that I use " $y=(...)+n$ " instead of " $y=(...)$ " to create a vertical shift. Similarly a vertical stretch will result from directly multiplying y by a constant, so instead of using " $y=(...)$ " we use " $y=a(...)$ ". And, that's what causes the order of operations to matter here. (Incidentally, one can multiply x directly by a constant to effect a horizontal stretch, replacing the " x " by " bx ", but for some reason we don't mention this. A more interesting question might be how can we combined a horizontal stretch with a horizontal shift to effect an equivalent vertical stretch.)

At any rate, with this preceding discussion in mind, let's just write them out and get them in standard form since there are only 6.

- i. (a) $\rightarrow$ (b) $\rightarrow$ (c): $y=ax^2 \rightarrow y=a(x+m)^2 \rightarrow y=a(x+m)^2+n$
- ii. (a) $\rightarrow$ (c) $\rightarrow$ (b): $y=ax^2 \rightarrow y=ax^2+n \rightarrow y=a(x+m)^2+n$
- iii. (b) $\rightarrow$ (a) $\rightarrow$ (c): $y=(x+m)^2 \rightarrow y=a(x+m)^2 \rightarrow y=a(x+m)^2+n$
- iv. (b) $\rightarrow$ (c) $\rightarrow$ (a): $y=(x+m)^2 \rightarrow y=(x+m)^2+n \rightarrow y=a[(x+m)^2+n]=a(x+m)^2+an$
- v. (c) $\rightarrow$ (a) $\rightarrow$ (b): $y=x^2+n \rightarrow y=a(x^2+n)=ax^2+an \rightarrow y=a(x+m)^2+an$
- vi. (c) $\rightarrow$ (b) $\rightarrow$ (a): $y=x^2+n \rightarrow y=(x+m)^2+n \rightarrow y=a[(x+m)^2+n]=a(x+m)^2+an$

So, the first three are equivalent and the last three are equivalent.

[Top](#)

Problem 263:

Consider a square with side of length x . A rectangle with the same perimeter as this square will have to have shorter sides of length $x-h$ and longer sides of length $x+h$ for some (positive) h if it is to have the same perimeter as the square. The area of the square is just the length of a side squared: x^2 . The area of a rectangle with the same perimeter is the product of the lengths of a short side and a long side: $(x-h)(x+h)=x^2-xh+xh-h^2=x^2-h^2$. But, $x^2-h^2 < x^2$ since $h^2 > 0$ for all h . Thus, the area of a square is always greater than the area of a rectangle with the same perimeter.

[Top](#)

Problem 264:

Suppose a rectangle with the same area as a square had a smaller perimeter. Then, another square with the same perimeter as the rectangle has an area greater than the rectangle by the preceding problem. But, that square with a smaller perimeter than the first square would then have an area greater than the first square which is impossible. (Suppose the length of a side of the first square is x and the length of a side of the second square is y . The perimeter of a square is just 4 times the length of a side, and so by hypothesis, $4x > 4y$ or $x > y$. But, the area of the second square is greater than the area of the rectangle which has the same area as the first square so that $y^2 > x^2$ since the area of a square is just the length of a side squared. In other words, $x > y$ and $y^2 > x^2$ which is a contradiction.)

Then there can be no rectangle with the same area as a square but with a smaller perimeter. Or, in other words, a square has the minimum perimeter of all rectangles having the same area.

[Top](#)

Problem 267:

- (a) $(x^2 - i)(x^2 + i) = x^4 - ix^2 + ix^2 - i^2 = x^4 - (-1) = x^4 + 1 = 0$ can have no (real) solutions.
- (b) $x^4 = 0$ only has $x = 0$ as a solution.
- (c) $(x^2 - 1)^2 = (x^2)^2 - 2(x^2) + 1 = x^4 - 2x^2 + 1 = 0$ as solutions $x = \pm 1$ because $(x - 1)^2 = 0$ has only the one solution $x = 1$.
- (d) $x^2(x - 1)(x + 1) = x^2(x^2 - x + x - 1) = x^2(x^2 - 1) = x^4 - x^2 = 0$ has solutions at (and only at) $x = 0, 1, -1$.
- (e) $(x - 1)(x + 1)(x - 2)(x + 2) = (x^2 - x + x - 1)(x^2 - 2x + 2x - 4) = (x^2 - 1)(x^2 - 4) = x^4 - x^2 - 4x^2 + 4 = x^4 - 5x^2 + 4 = 0$ has solutions at $x = 1, -1, 2, -2$ by design. (Also, note the trick seen previously of using the product of conjugates here as well as in (a) and in (d).)
- (f) A fourth degree polynomial can have at most four distinct roots.

[Top](#)

Problem 269:

The expression can, at best, be factored into $a(x^4 - r_1^4)(x^4 - r_2^4) = 0$ which has at most four solutions: $x = \pm r_1, \pm r_2$. It is easy enough to imagine such a scenario: $(x^4 - 1)(x^4 - 16) = x^8 - 17x^4 + 16 = 0$. Also, we can arrange for 3 solutions with $x^4(x^4 - 1) = 0$, 2 solutions with $(x^4 - 1)^2 = 0$, 1 solution with just $x^4 = 0$, and no solutions with $x^4 + 1 = 0$. Of course, if all the coefficients are zero, we just have $0 = 0$ which is true for all x . So, our answer is 0, 1, 2, 3, 4, or infinitely many.

[Top](#)

Problem 271:

Raise 1.2 to the 10th power and compare the result to 2. In order to calculate $(1.2)^{10}$ without a calculator, let's use Pascal's Triangle to calculate $12^{10} = (10+2)^{10} = 10^{10} + 10 \times 10^9 \times 2^1 + C = 3 \times 10^{10} + C$ where C is some extra terms that add up to something greater than zero. Thus, $12^{10} > 3 \times 10^{10}$ which means that $(1.2)^{10} > 3 > 2$, so that 1.2 is greater than the tenth root of 2.

[Top](#)

Problem 272:

Consider $(0.9995)^2 = (1-0.0005)^2 = 1 - 2 \times 1 \times (0.0005) + (0.0005)^2 = 0.999 + 0.0000025 = 0.999001$. Clearly, if I multiply by 0.9995 again, I'll get below the target of 0.999. So whatever the 7th root of 0.999 is, it must be greater than 0.9995. On the other hand, if the 7th root was greater than 1, then multiplying it by itself seven times would only yield a larger number, which is already larger than 1. But, it must multiply out to a number smaller than 1 (namely 0.999). So, the 7th root must be smaller than 1 and larger than 0.9995. Then, rounded to the third decimal place, it be equal to 1.000.

[Top](#)

Problem 273:

Raise each number by the least common multiple of 2 and 3 (6). Root 2 raised to the 6th power is 2 cubed is 8. The cube root of three raised to the 6th power is 3 squared is 9 which is greater than 8. Thus the square root of 2 is less than the cube root of 3.

[Top](#)

Problem 274:

The fourth root of four is just the square root of the square root of 4 or, in other words, the square root of 2 which we just saw in the previous problem was less than the cube root of 3.

[Top](#)

Problem 275:

The square root of the square root *is* the fourth root. (They are equal – a number twice squared is the same as that number raised to the fourth power.)

[Top](#)

Problem 278:

Let x be the n^{th} root of a/b , y be the n^{th} root of a , and z be the n^{th} root of b . Then, $x^n = a/b$, $y^n = a$, and $z^n = b$, so that $x^n = y^n/z^n = (y/z)^n$. Therefore, $x = y/z$. (Or, just substitute $1/b$ for b in problem # 277.)

[Top](#)

Problem 279:

This result is corollary to the fact that $1^n = 1$ and the previous problem.

[Top](#)

Problem 284:

By problem # 283, the mn^{th} root is the m^{th} root of the n^{th} root. For non-negative a , the n^{th} root of a^n is a . Thus, this result is corollary to problem # 283.

[Top](#)

Problem 285:

This result is corollary to problem #277 and the fact that, for non-negative a , the n^{th} root of a^n is a .

[Top](#)

Problem 286:

He's just asking you to write the properties down in terms of p and q

1. $(ab)^p = a^p b^p$

$$2. (a/b)^p = a^p/b^p$$

3. In particular, since $1^n = 1$ so the n^{th} root of 1 is 1, and letting $a=1$ and $b=a$ in the above statement, $(1/a)^p = 1/a^p$.

$$4. (abc)^p = a^p b^p c^p$$

$$5. a^{pq} = (a^p)^q$$

That should more or less do the trick of restating as well as deriving the various facts for rational numbers.

[Top](#)

Problem 288:

Mainly, the n^{th} root of a^n is a (for nonnegative a). So, in particular, if p in a^p is not in lowest terms, it can be expressed as $(mr)/(nr)$ for some integers m , n , and r . Thus, $a^p = a^{(mr)/(nr)} = (a^{1/nr})^{mr} = ((a^{1/n})^{1/r})^m = ((a^{1/n})^m)^{1/r} = (a^{m/n})^{1/r}$. And the last expression is just the r^{th} root of $(a^{m/n})^r$ which is $a^{m/n}$. Therefore, $a^p = a^{m/n}$ or, in other words, we get the same result if we reduce p some.

[Top](#)

Problem 290:

$$(ab)^{(m/n)} = [(ab)^{1/n}]^m = (a^{1/n} b^{1/n})^m = (a^{1/n})^m (b^{1/n})^m = a^{m/n} b^{m/n}$$

[Top](#)

Problem 292:

Back in problem # 272, I wasn't being rigorous so much as just trying to get the answer at the time. Now we have to look at this matter a little more carefully. Consider a number $a=1+x$ for some $x>0$. Then,

$$a^n = (1+x)^n = 1^n x^0 + n 1^{n-1} x^1 + C(x) = 1 + nx + C(x)$$

where $C(x)$ is a polynomial in x with coefficients corresponding to Pascal's Triangle. Actually, it takes the Binomial Theorem to really know what those coefficients are and that $C(x)>0$ if $x>0$. Nevertheless, hopefully it is sufficiently rigorous for this book to just observe this fact and note that, then, $a^n > 1$. Alternatively, we might use the Principle of Mathematical Induction to prove that if $a>1$, then $a^n > 1$ for all n . Suppose $a^{n-1} > 1$ and $a>1$. Then, $a^n = (1+x)a^{n-1} = a^{n-1} + xa^{n-1} > a^{n-1} > 1$. So,

perhaps it is intuitively clear that since we start out with a number greater than 1, all we can do is get even bigger by multiplying it by itself.

Alternatively, suppose that $0 < a < 1$ so that $a = 1 - x$ for some $x > 0$. Then if $a^{n-1} < 1$, multiplying by a again would result in $a^n = (1-x)a^{n-1} = a^{n-1} - xa^{n-1} < a^{n-1} < 1$. While this argument isn't really mathematical induction with all the bells and whistles, this is essentially what it all boils down to. We start out with a number less than one, and then multiplying by it just gets us a smaller number. Once we have these two facts established, everything else is an extension of them. Also, note that $1^n = 1$ and if $a^n = 1$ then $a = 1$ (or in other words, the n^{th} root of 1 is 1).

Suppose $a^{1/n} < 1$. Then, $a = (a^{1/n})^n < 1$. Or, contrapositively, if $a > 1$, then $a^{1/n} > 1$ (recalling that the n^{th} root of 1 is 1). Conversely, if $a^{1/n} > 1$, then $a = (a^{1/n})^n > 1$, so that if $a < 1$, then $a^{1/n} < 1$. In both cases, I use the fact that if A implies B, then not B implies not A. For instance, if $a^{1/n} < 1$ implies that $a < 1$, then $a \geq 1$ implies that $a^{1/n} \geq 1$. And, since equality only occurs in the case of $a = a^{1/n} = 1$, I can simply conclude that $a > 1$ implies that $a^{1/n} > 1$.

So, finally, let $a > 1$ and let p and q be two rational numbers such that $p < q$. Then, $q - p$ is some rational number greater than zero which can be represented as $q - p = m/n$ for some positive integers m and n . Then, $a^{q-p} = a^{m/n} = (a^{1/n})^m > 1$ based on the preceding discussion (since $b = a^{1/n} > 1$ implies that $(a^{1/n})^m = b^m > 1$). But, by previous problems, $1 < a^{q-p} = a^{q+(-p)} = a^q a^{(-p)} = a^q / a^p$. Then, since $a^p > 0$, $a^p < a^q$. To recap, if $a > 1$ and p and q be two rational numbers such that $p < q$, then $a^p < a^q$. Conversely, if $a < 1$ and p and q be two rational numbers such that $p < q$, then $a^{q-p} = a^{m/n} = (a^{1/n})^m < 1$ which implies that $1 > a^q / a^p$ so that $a^p > a^q$.

[Top](#)

Problem 299:

$$2000^{1000} = 2^{1000} 1000^{1000} < 1000^{1000} 1000^{1000} = (1000^{1000})^2 = 1000^{2000}$$

[Top](#)

Problem 300:

This problem is the essence of the fact that the Harmonic Series converges which is a huge fact and a big deal in mathematics! At any rate, there is a trick to it – bound the series by another series that is easier to analyze. So, for each part of this problem we are going to look at two different series as follows:

1	≥ 1	≥ 1
1/2	$\geq 1/2$	$\geq 1/2$
1/2	$\geq 1/3$	$\geq 1/4$
1/4	$\geq 1/4$	$\geq 1/4$
1/4	$\geq 1/5$	$\geq 1/8$
1/4	$\geq 1/6$	$\geq 1/8$

$$\begin{array}{ccc} 1/4 & \geq 1/7 & \geq 1/8 \\ 1/8 & \geq 1/8 & \geq 1/8 \end{array}$$

The numbers in the middle are the terms in the Harmonic Series which is the series that we are looking at in this problem. Our series is less than the series consisting of the terms in the first column and greater than the series consisting of the terms in the third column. But, the series in the first column is actually easy to add up:

$$\begin{aligned} &1 + 1/2 + 1/2 + 1/4 + 1/4 + 1/4 + 1/4 + \dots \\ &= 1 + (1/2 + 1/2) + (1/4 + 1/4 + 1/4 + 1/4) + \dots \\ &= 1 + 1 + 1 + \dots \end{aligned}$$

Similarly, the series in the third column adds up as follows:

$$\begin{aligned} &1 + 1/2 + 1/4 + 1/4 + 1/8 + 1/8 + 1/8 + 1/8 + \dots \\ &= 1 + 1/2 + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8) + \dots \\ &= 1 + 1/2 + 1/2 + 1/2 + \dots \end{aligned}$$

So, the second part of the problem is easy enough to show, then. No matter what number we are given, we can simply add up enough terms in the Harmonic Series so that the sum is greater than the sum of the terms of the series in the third column, and if we have selected enough terms, that sum will just be 1 plus a half added to itself sufficiently many times to exceed the number we were originally given. Figuring out just how many terms that is, on the other hand, is along the lines of what is required in the first part of the problem.

What we are adding up in both cases, but in particular in the first column are powers of 2. Each time to get our next 1, we add up 2^n terms consisting of $1/2^n$, thus getting the next 1 as $2^n \times (1/2^n)$. We need to figure out what it all adds up to if we go out 1,000,000 terms. It will be 1 added to itself n times where n is such that $2^0 + 2^1 + 2^2 + \dots + 2^n = 1,000,000$. In other words, we get exactly 20 if we add up the first series'

$$\begin{aligned} &2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10} + 2^{11} + 2^{12} + 2^{13} + 2^{14} + 2^{15} + 2^{16} + 2^{17} + 2^{18} + 2^{19} + 2^{20} \\ &= 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1,024 + 2,048 + 4,096 + 8,192 \\ &\quad + 16,384 + 32,768 + 65,536 + 131,072 + 262,144 + 524,288 + 1,048,576 \\ &> 1,000,000. \end{aligned}$$

In fact, the sum is much greater than 1,000,000. In other words, if all we do is sum up the first 1,000,000 terms of the first series we don't get anywhere near 20, and since the Harmonic Series that we are actually interested in is less than the first series, the sum of *its* first million terms isn't anywhere close to 20 either.

And, while we do already know that sooner or later the Harmonic Series will add up to get over 20, let's go ahead and try to find an upper bound on how many terms we must add up to get there. We know that the third series bounds it from below so the Harmonic Series will have to add to 20 long before the third series does. The third series is 1 plus $1/2^n$ added to itself 2^{n-1} times for each successive n , thus getting another $2^{n-1}(1/2^n) = 1/2$ each time. So, we need $2 \times 19 = 38$ halves to get us to 20 in the third series. Using a result from some previous problems:

$$1 + 2 + 2^2 + \dots + 2^{37} = (2^{38} - 1) / (2 - 1) = 2^{38} - 1 \approx 2.75 \times 10^{11}$$

So, if you were locked in the closet until you added up sufficiently many terms of the Harmonic Series to reach 20, you can be assured to know that it shouldn't take more than 2.75×10^{11} terms! :o)

[Top](#)

Problem 302:

Just solve for a in $(1+a)/2=7$: $1+a=14$ so that $a=13$.

[Top](#)

Problem 303:

Just solve for a in $\sqrt{1 \times a}=7$: $a=7^2=49$.

[Top](#)

Problem 304:

- (a) The perimeter of the rectangle is $2a+2b$. If the square has side of length c , then its perimeter is $4c$. Then, $4c=2a+2b$, so that $c=(a+b)/2$, the arithmetic mean of the rectangle's sides.
- (b) The area of the rectangle is ab . If the square has side of length c , then its area is c^2 . Then, $c^2=ab$, so that $c=\sqrt{ab}$, the geometric mean of the rectangle's sides.

[Top](#)

Problem 310:

The maximum possible area is achieved if we enclose a square piece of land. Thus, the land will have to have a side of length $120\text{m}/4 = 30\text{m}$ and so would have area $(30\text{m})^2=900\text{m}^2$.

[Top](#)

Problem 313:

The square in the middle clearly has side length $a-b$ and so area $(a-b)^2 = a^2 - 2ab + b^2$. In the algebraic proof, we finally noticed that the inequality between the arithmetic and geometric mean was equivalent to the fact that $(a-b)^2 > 0$.

[Top](#)

Problem 315:

Suppose that $a_1 + \dots + a_n = S$. Then, $S = a_1 + \dots + a_n \geq n(a_1 \times \dots \times a_n)^{1/n}$, so that $a_1 \times \dots \times a_n \leq (S/n)^n$. If $a_i = a_j = a$ for all i and j , then $a_1 \times \dots \times a_n = a^n$ and $S = a_1 + \dots + a_n = na$. Thus, $a_1 \times \dots \times a_n = (S/n)^n$ when $a_i = a_j$ for all i and j which, in other words, is, by the inequality, the maximum value the product can attain.

[Top](#)

Problem 316:

Suppose that $a_1 \times \dots \times a_n = P$. Then, $nP^{1/n} = n(a_1 \times \dots \times a_n)^{1/n} \leq (a_1 + \dots + a_n)$. If $a_i = a_j = a$ for all i and j , then $a_1 \times \dots \times a_n = a^n$ and $a_1 + \dots + a_n = na$. Thus, $nP^{1/n} = n(a^n)^{1/n} = na$ and $(a_1 + \dots + a_n) = na$, so that $nP^{1/n} = (a_1 + \dots + a_n)$ when $a_i = a_j$ for all i and j which, in other words, is, by the inequality, the minimum value the sum can attain.

[Top](#)

Problem 318:

Let a, b, c , and d be any real numbers such that $(abcd)^{1/4} = (a+b+c+d)/4$. Let $x = (a+b)/2$ and $y = (c+d)/2$. We already know that $\sqrt{xy} \leq (x+y)/2$ or that $xy \leq (x+y)^2/4$. Furthermore, $xy = [(a+b)/2][(c+d)/2] \geq [\sqrt{ab}][\sqrt{cd}] = \sqrt{abcd}$ by the same original inequality. Thus, $\sqrt{abcd} \leq (x+y)^2/4$, so that

$$(a+b+c+d)/4 = (abcd)^{1/4} = \sqrt{[\sqrt{abcd}]} \leq \sqrt{xy} \leq (x+y)/2 = [(a+b)/2 + (a+b)/2]/2 = (a+b+c+d)/4$$

Since the far left is exactly the same as the far right, everything in between must be equal. In particular, $\sqrt{xy} = (x+y)/2$ which can only happen if $x=y$ based on our previous theorem. Then,

$$(abcd)^{1/4} = (x+y)/2 = x = (a+b)/2 \geq \sqrt{ab}.$$

Quadrupling both sides and dividing by ab , we have that $cd \geq ab$. But, by the same token,

$$(abcd)^{1/4} = (x+y)/2 = y = (c+d)/2 \geq \sqrt{cd}.$$

And so, quadrupling both sides and dividing by cd , we have that $cd \leq ab$, as well. Then, $cd = ab$. In that case, we have that $(a+b)/2 = (a+b+c+d)/4 = (abcd)^{1/4} = (ab)^{1/2}$ which implies that $a = b$, by our prior theorem. And similarly, $(c+d)/2 = (a+b+c+d)/4 = (abcd)^{1/4} = (cd)^{1/2}$ implies that $c = d$, by our prior theorem. Finally since $x = y$, $a + b = c + d$ so that $2b = 2c$ or $b = c$. Then, $a = b = c = d$.

[Top](#)

Problem 321:

Use the same trick as in the preceding problem. Consider the respective means between the following 8 numbers: a, b, c, d, e, f, g , and $(abcdefg)^{1/7}$. We know that $[abcdefg(abcdefg)^{1/7}]^{1/8} \leq [a+b+c+d+e+f+g+(abcdefg)^{1/7}]/8$ by problem # 319. But,

$$[abcdefg(abcdefg)^{1/7}]^{1/8} = [(abcdefg)^{7/7}(abcdefg)^{1/7}]^{1/8} = [(abcdefg)^{8/7}]^{1/8} = (abcdefg)^{1/7}$$

and so

$$\begin{aligned} (abcdefg)^{1/7} &\leq [a+b+c+d+e+f+g+(abcdefg)^{1/7}]/8 \rightarrow \\ 8(abcdefg)^{1/7} &\leq a+b+c+d+e+f+g+(abcdefg)^{1/7} \rightarrow \\ 7(abcdefg)^{1/7} &\leq a+b+c+d+e+f+g \rightarrow \\ (abcdefg)^{1/7} &\leq (a+b+c+d+e+f+g)/7. \end{aligned}$$

[Top](#)

Problem 322:

Keep using the same trick. Consider the respective means between the following 7 numbers: a, b, c, d, e, f , and $(abcdef)^{1/6}$. We know that $[abcdef(abcdef)^{1/6}]^{1/7} \leq [a+b+c+d+e+f+(abcdef)^{1/6}]/7$ by problem # 321. But,

$$[abcdef(abcdef)^{1/6}]^{1/7} = [(abcdef)^{6/6}(abcdef)^{1/6}]^{1/7} = [(abcdef)^{7/6}]^{1/7} = (abcdef)^{1/6}$$

and so

$$\begin{aligned} (abcdef)^{1/6} &\leq [a+b+c+d+e+f+(abcdef)^{1/6}]/7 \rightarrow \\ 7(abcdef)^{1/6} &\leq a+b+c+d+e+f+g+(abcdef)^{1/6} \rightarrow \\ 6(abcdef)^{1/6} &\leq a+b+c+d+e+f \rightarrow \\ (abcdef)^{1/6} &\leq (a+b+c+d+e+f)/6. \end{aligned}$$

[Top](#)

Problem 323:

To prove that the arithmetic mean is greater than the geometric mean for all $n=2^k$ for some k , you should actually use induction on k . Similar to other such problems let's just convince ourselves by roughly doing the inductive step. Suppose the arithmetic mean was greater than the geometric mean for 2^k numbers. Then, consider 2^{k+1} numbers and group them in two groups of 2^k numbers. The first group's geometric mean is smaller than its arithmetic mean as is the second group's, so the square root of the product of the two geometric means is less than the square root of the product of the two groups arithmetic means. The square root of the product of the geometric means is just the geometric mean of the entire 2^{k+1} numbers. And, the square root of the product of the arithmetic means is just the arithmetic mean of the arithmetic means which, by the original theorem about two numbers, is less than the arithmetic mean of the arithmetic means. Finally, the arithmetic mean of the arithmetic means is just the arithmetic mean of the whole 2^{k+1} set of numbers. So, if the inequality holds for 2^k numbers, then it also holds for 2^{k+1} numbers. That it holds for 2 was our first theorem, and so it has to hold for all the rest of the 2^k numbers of values. (That's basically a proof by induction in words.)

Consider the set of all numbers n of values for which the inequality of the arithmetic and geometric mean holds. Suppose for some n the inequality did not hold. There exists k such that $2^k > n$. Since the inequality holds for 2^k , there exists an integer m with $n \leq m < 2^k$ such that the inequality holds for $m+1$ but not for m . But, following a similar proof to the preceding problem, we can show that the inequality must hold for m if it holds for $m+1$. Consider the following $m+1$ values: $a_1, a_2, \dots, a_m$, and $(a_1 a_2 \dots a_m)^{1/m}$.

We know that $[a_1 a_2 \dots a_m (a_1 a_2 \dots a_m)^{1/m}]^{1/(m+1)} \leq [a_1 + a_2 + \dots + a_m + (a_1 a_2 \dots a_m)^{1/m}] / (m+1)$ by hypothesis. But,

$$[a_1 a_2 \dots a_m (a_1 a_2 \dots a_m)^{1/m}]^{1/(m+1)} = [(a_1 a_2 \dots a_m)^{m/m} (a_1 a_2 \dots a_m)^{1/m}]^{1/(m+1)} = [(a_1 a_2 \dots a_m)^{(m+1)/m}]^{1/(m+1)} = (a_1 a_2 \dots a_m)^{1/m}$$

and so

$$\begin{aligned} (a_1 a_2 \dots a_m)^{1/m} &\leq [a_1 + a_2 + \dots + a_m + (a_1 a_2 \dots a_m)^{1/m}] / (m+1) \rightarrow \\ (m+1)(a_1 a_2 \dots a_m)^{1/m} &\leq a_1 + a_2 + \dots + a_m + (a_1 a_2 \dots a_m)^{1/m} \rightarrow \\ m(a_1 a_2 \dots a_m)^{1/m} &\leq a_1 + a_2 + \dots + a_m \rightarrow \\ (a_1 a_2 \dots a_m)^{1/m} &\leq (a_1 + a_2 + \dots + a_m) / m \end{aligned}$$

But, this result contradicts the hypothesis that the inequality did not hold for m . Thus, there can be no such n for which the inequality does not hold.

[Top](#)

Problem 324:

I'm imagining he's thinking we can go back through the previous argument and consider what equality would imply. I, myself, would use mathematical induction. I don't think that going back through the previous argument is all that rigorous. At any rate, let's just do the inductive step. Suppose that, for n terms, whenever the arithmetic mean equals the geometric mean, then the terms are all equal. Now

suppose that the arithmetic mean equals the geometric mean for $n+1$ terms: $a_1, a_2, \dots, a_{n+1}$. Let a be the n^{th} root of $a_1 a_2 \dots a_n$, and suppose that the geometric mean of a and a_{n+1} is the same as the arithmetic mean (i.e. $\sqrt[n+1]{a a_{n+1}} = (a + a_{n+1})/2$). Then, $a = a_{n+1}$ by the original theorem, so that the $n+1$ root of $a_1 a_2 \dots a_n a_{n+1}$ is the $n+1$ root of $(a^n a_{n+1}) = a_{n+1}^{n+1}$ which is just a_{n+1} . Then, since the means are equal $(a_1 + a_2 + \dots + a_{n+1})/(n+1) = a_{n+1}$, so that $a_1 + a_2 + \dots + a_{n+1} = (n+1)a_{n+1} = n a_{n+1} + a_{n+1}$, and so $a_1 + a_2 + \dots + a_n = n a_{n+1}$. So, dividing by n , $(a_1 + a_2 + \dots + a_n)/n = a_{n+1} = a$, so that the geometric mean of the first n terms equals the arithmetic mean. But, the original (inductive) hypothesis, was that if the arithmetic mean equals the geometric mean for n terms, then the terms must be all equal, so that $a_1 = a_2 = \dots = a_n = a_{n+1} = a$.

That is, if the arithmetic mean equaling the geometric mean of n terms implies that the terms are all equal, then the arithmetic mean equaling the geometric mean of $n+1$ terms implies that all the terms are equal in that case, too. We already know by independent means that such equality for 2 terms implies the two terms are equal to each other, and so we must know that the equality of the means only holds for any number of terms only if the terms are all equal. If there was some number of terms k it didn't work for, then it couldn't work for one fewer number of terms since otherwise, it would have to work for k terms too, by the above argument. Similarly, it couldn't work for one fewer terms than that and so on all the way back to 2 terms. But we know it does work for 2 terms, so there can be no such k terms it doesn't work for.

(That's kind of a poorly presented proof by Mathematical Induction, but one that hopefully convinces you of the necessity of its conclusion.)

[Top](#)

Problem 326:

Consider the product of the terms in the sum on the left hand side:

$$(a_1/a_2)(a_2/a_3)\dots(a_{n-1}/a_n)(a_n/a_1) = (a_1 a_2 \dots a_n)(a_2 \dots a_n a_1) = 1$$

Since the product of the terms is 1, the geometric mean of those terms is one. Since the arithmetic mean is greater than the geometric mean,

$$[(a_1/a_2) + (a_2/a_3) + \dots + (a_{n-1}/a_n) + (a_n/a_1)]/n \geq 1 \rightarrow \\ (a_1/a_2) + (a_2/a_3) + \dots + (a_{n-1}/a_n) + (a_n/a_1) \geq n$$

[Top](#)

Problem 327:

$$\sqrt[3]{ab^2} = \sqrt[3]{a b b} \leq (a + b + b)/3 = (a + 2b)/3$$

[Top](#)

Problem 328:

$$(a+b)/2 \geq \sqrt{ab} \rightarrow a+b \geq 2\sqrt{ab}$$

Equality holds only if $a=b$, so that $a+b$ attains its minimum value only when $a=b$. But, if $a=b$ and $1=ab^2=a^3$, then $a=b=1$, so that $a+b=2$.

[Top](#)

Problem 329:

$$\sqrt[6]{a^3 b^2 c} = [\sqrt[6]{a}] [\sqrt[6]{b^2}] [\sqrt[6]{c^3}] = \sqrt[6]{a^3 b^2 c^3} \leq (a+b+b+c+c+c)/6 = (a+2b+3c)/6$$

[Top](#)

Problem 330:

$$(a+2b+3c)/3 \geq \sqrt[3]{(a(2b)(3c))} = (\sqrt[3]{6}) [\sqrt[3]{abc}] \rightarrow (a+2b+3c)/[3(\sqrt[3]{6})] \geq \sqrt[3]{abc}$$

[Top](#)

Problem 332:

Since all the terms are equal, the arithmetic mean of 11 terms of $(1+1/10)$ is equal to its geometric mean, so that $(1+1/10)^{11}$ is just equal to the arithmetic mean raised to the 11th power. Following the hint, $(1+1/11)^{12} = (1+2/11+1/121)(1+1/11)^{10}$ is the geometric mean of 11 terms raised to the 11th power. Consider the sum of these 11 terms: $1+2/11+1/121+10(1+1/11)=11+12/11+1/121=12+1/11+1/121$. The arithmetic mean is greater than the geometric mean, so that $(1+1/11)^{12} = (1+2/11+1/121)(1+1/11)^{10} \leq [(12+1/11+1/121)/12]^{11} = [1+1/(11 \times 12)+1/(121 \times 12)]^{11}$. But, the sum of the 11 terms of the product of $(1+1/10)$'s is $11(1+1/10)=11+11/10=12+1/10$, and so its arithmetic mean is $(12+1/10)/11 = 12/10+1/(10 \times 11) = 1+1/10+1/(10 \times 11) > 1+1/(11 \times 12)+1/(121 \times 12)$ since each of the three terms in the sum are greater than each of the three terms of the other sum, respectively. Therefore, since its arithmetic mean equals its geometric mean:

$$(1+1/10)^{11} = [1+1/10+1/(10 \times 11)]^{11} > [1+1/(11 \times 12)+1/(121 \times 12)]^{11} \geq (1+1/11)^{12}$$

[Top](#)

Problem 333:

$$(1+10)^{10} < (1+1/11)^{11} < (1+1/11)^{12} < (1+1/10)^{11}$$

[Top](#)

Problem 334:

The geometric mean is $\sqrt{a^2 b^2} = ab$.

[Top](#)

Problem 336:

Just mimic the preceding proof (in the book).

$$\begin{aligned}\sqrt{(a^2+b^2)/2} &= (a+b)/2 \rightarrow \\ (a^2+b^2)/2 &= [(a+b)/2]^2 = (a^2+2ab+b^2)/4 \rightarrow \\ 2(a^2+b^2) &= a^2+2ab+b^2 \rightarrow \\ a^2+b^2 &= 2ab \rightarrow \\ a^2-2ab+b^2 &= (a-b)^2 = 0 \rightarrow \\ a &= b\end{aligned}$$

Thus, if the quadratic mean equals the arithmetic mean of two terms, then the terms must be equal.

[Top](#)

Problem 337:

The arithmetic mean sits in the middle:

$$\sqrt{ab} \leq (a+b)/2 \leq \sqrt{(a^2+b^2)/2}$$

[Top](#)

Problem 339:

Suppose for the moment that they are equal:

$$\begin{aligned}
(a+b+c)/3 &= \sqrt{[(a^2+b^2+c^2)/3]} \rightarrow \\
[a+(b+c)]^2/9 &= [(a^2+b^2+c^2)/3] \rightarrow \\
a^2+2a(b+c)+(b+c)^2 &= 3(a^2+b^2+c^2) \rightarrow \\
a^2+2ab+2ac+b^2+2bc+c^2 &= 3a^2+3b^2+3c^2 \rightarrow \\
0 &= 2a^2+2b^2+2c^2-2ab-2ac-2bc
\end{aligned}$$

We've actually seen this or something like it before. It is going to actually be something like the following in disguise:

$$(a-b)^2+(b-c)^2+(c-a)^2 = a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ca+a^2 = 2a^2+2b^2+2c^2-2ab-2ac-2bc$$

So, continuing from above, if the two means are equal then we have that

$$(a-b)^2+(b-c)^2+(c-a)^2 = 0$$

The sum of nonnegative terms can only be zero if each term is zero. So, $a=b=c$ if the means are equal. While this wasn't what we were supposed to prove, it is the companion to it (what happens when the means are equal) and it also shows us the way to prove it. Since if the terms are *not* equal, we can just proceed backwards along the first derivation only this time with an inequality:

$$\begin{aligned}
0 &\leq (a-b)^2+(b-c)^2+(c-a)^2 \rightarrow \\
0 &\leq 2a^2+2b^2+2c^2-2ab-2ac-2bc \rightarrow \\
a^2+2ab+2ac+b^2+2bc+c^2 &\leq 3a^2+3b^2+3c^2 \rightarrow \\
a^2+2a(b+c)+(b+c)^2 &\leq 3(a^2+b^2+c^2) \rightarrow \\
[a+(b+c)]^2/9 &\leq [(a^2+b^2+c^2)/3] \rightarrow \\
(a+b+c)/3 &\leq \sqrt{[(a^2+b^2+c^2)/3]}.
\end{aligned}$$

[Top](#)

Problem 340:

a) We have that

$$\begin{aligned}
1 &= 2/2 = (a+b)/2 \leq \sqrt{[(a^2+b^2)/2]} \rightarrow \\
1 &\leq (a^2+b^2)/2 \rightarrow \\
2 &\leq (a^2+b^2)
\end{aligned}$$

So, the minimum value of the sum of their squares is 2. And, this is attained, for instance, if the two numbers are $a=b=1$.

b) Similarly,

$$\begin{aligned}
2/3 &= 2/3 = (a+b+c)/3 \leq \sqrt{[(a^2+b^2+c^2)/3]} \rightarrow \\
4/9 &\leq (a^2+b^2+c^2)/3 \rightarrow \\
4/3 &\leq (a^2+b^2+c^2)
\end{aligned}$$

So, the minimum value of the sum of their squares is $4/3$. And, this is attained, for instance, if the three numbers are $a=b=c=2/3$.

[Top](#)